

Subject: Math - round II

Time limit: 150 minutes

Q1 (3.5pts) 1) Solve system of equation

$$\begin{cases} x + y = \sqrt{x + 3y} \\ x^2 + y^2 + xy = 3 \end{cases}$$

2) For $a, b \in \mathbb{R}^+$ s.t. $ab + a + b = 1$, prove

$$\frac{a}{1+a^2} + \frac{b}{1+b^2} = \frac{1+ab}{\sqrt{1+(1+a^2)(1+b^2)}}$$

Q2 (2.5pts) 1) For $p, q \in \mathbb{N}^+$ s.t.

$$p(p-1) = q(q^2-1) \quad (*)$$

- Prove that there exists $k \in \mathbb{N}^+$ that satisfies $p-1 = kq, q^2-1 = kp$
- Find all p, q that satisfy (*)

2) For $a, b, c \in \mathbb{R}^+$ s.t $ab + bc + ca + abc = 2$, find the max of

$$M = \frac{a+1}{a^2+2a+2} + \frac{b+1}{b^2+2b+2} + \frac{c+1}{c^2+2c+2}$$

Q3 (3pts) Let ABC be an acute triangle with $AB < AC$ (length). E, F are midpoints of AC, AB respectively. The midperpendicular of EF crosses BC at D . Suppose there exists P inside \widehat{EAF} (notation for angle, i.e., $\angle EAF$) and outside of the triangle AEF s.t. $\widehat{PEC} = \widehat{DEF}$ and $\widehat{PFB} = \widehat{DFE}$. PA crosses the circumscribed circle of PEF at $Q \neq P$

- 1) Prove $\widehat{EQF} = \widehat{BAC} + \widehat{DEF}$
- 2) The tangent at P of the circumscribed circle of PEF crosses CA, AB at M, N respectively. Prove that four points C, M, B, N lie on the same circle. Call this circle (K)
- 3) Prove that (K) is tangent to the circumscribed circle of AEF

Q4 (1pt) Let $n \in \mathbb{N}^+, n \geq 5$. Consider a convex polygon with n edges. We want to draw several diagonals so that they divide the convex polygon into k regions, each being a convex pentagon (any two regions do not overlap except for possibly the edges)

- 1) Prove that we can do this for $n = 2018, k = 672$
- 2) For $n = 2017, k = 672$, can we do this? Explain.