EXAMPLE 3

Use Eq. (13) to find the linear system corresponding to the pendulum equations (8) near the origin; near the critical point $(\pi, 0)$.

In this case we have, from Eq. (8),

$$F(x,y) = y, \qquad G(x,y) = -\omega^2 \sin x - \gamma y; \tag{15}$$

since these functions are differentiable as many times as necessary, the system (8) is locally linear near each critical point. The derivatives of F and G are

$$F_x = 0, \qquad F_y = 1, \qquad G_x = -\omega^2 \cos x, \qquad G_y = -\gamma. \tag{16}$$

Thus, at the origin the corresponding linear system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{17}$$

which agrees with Eq. (9).

Similarly, evaluating the partial derivatives in Eq. (16) at $(\pi, 0)$, we obtain

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{18}$$

where $u = x - \pi, v = y$. This is the linear system corresponding to Eqs. (8) near the point $(\pi, 0)$.

We now return to the locally linear system (4). Since the nonlinear term $\mathbf{g}(\mathbf{x})$ is small compared to the linear term $\mathbf{Ax}$ when $\mathbf{x}$ is small, it is reasonable to hope that the trajectories of the linear system (1) are good approximations to those of the nonlinear system (4), at least near the origin. This turns out to be true in many (but not all) cases, as the following theorem states.

Theorem 9.3.2 Let r_1 and r_2 be the eigenvalues of the linear system (1) corresponding to the locally linear system (4). Then the type and stability of the critical point (0, 0) of the linear system (1) and the locally linear system (4) are as shown in Table 9.3.1.

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
r_1, r_2	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$r_1 = r_2 < 0$	PN or IN	Asymptotically stable	N or SpP	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$r_1 = i\mu, r_2 = -i\mu$	C	Stable	C or SpP	Indeterminate

Note: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.