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The point space of a frame

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In [3] we constructed a contravariant functor

$$\textbf{\textit{Top}} \longrightarrow \textbf{\textit{Frm}}$$

from spaces to frames. The object assignment sends a space S to its topology $\mathcal{O}S$, and the arrow assignment sends a map ϕ to the inverse image function $\phi^{\leftarrow}$ (restricted to the topologies). In this document we show that this functor is one half of a contravariant adjunction between the two categories. The object assignment in the other direction

$$A \longmapsto \mathsf{pt}(A)$$

sends a frame A to its point space $\mathsf{pt}(A)$. This is an important construction which enables quite a lot of point-sensitive topology, that is point set topology, to be done in a point-free way, that is using frames.

In Section 1 we first set up the point space and the associated adjunction in what may seem to be a rather *ad hoc* fashion. After that, in Section 2, we show that the adjunction is schizophrenically induced (by a rather trivial object). This explains much of the behaviour of the adjunction, and brings out many of its other features. You may prefer to read that section first before reading the *ad hoc* version. In Section 3 we give an entirely point-sensitive account of the construction. This, in fact, was the original version of the construction, and again you might want to read it before either of the first two sections.

We then look at two particular examples of the point-space construction. In Section 4 we look at the ideal completion of a poset, and in Section 5 we look at the Stone representation of distributive lattices. These two section can be read in either order.

The point space construction depends on attaching a set $\mathsf{pt}(A)$ of points to a frame A. In can happen that $\mathsf{pt}(A)$ is empty even though A is quite large and complicated. Section 6 gives an example of such an exotic frame. On the other hand, many frames have enough points to be a topology in disguise. Section 7 considers this aspect. This involves the use of variants of Zorn's Lemma. Again these last two section can be read in either order.

Contents

1 The point space construction

We wish to convert a frame A into a space. To do that we need to produce a set of points, and then furnish this with a topology. It turns out that the points we need are sitting inside A.

1.1 DEFINITION. Let A be a frame. An element $p \in A$ is $\wedge$-irreducible if $p \neq \top$ and

$$x \wedge y \leq p \Longrightarrow x \leq p \text{ or } y \leq p$$

for all $x, y \in A$.

Let $\mathsf{pt}(A)$ be the set of $\wedge$-irreducible elements of A. ■

Notice that a $\wedge$-irreducible element has a certain prime-like property. In fact, an element p is $\wedge$-irreducible exactly when the principal ideal $\downarrow p$ is prime. (We take a closer look at this and related matters in Section 2.)

We use $\mathsf{pt}(A)$ to carry a topology, and so we often refer to a $\wedge$-irreducible element of A as a point of A. But before we set up this topology let's get some idea of what $\mathsf{pt}(A)$ can look like.

1.2 LEMMA. *Let A be a frame.*

Each maximal element of A is $\wedge$-irreducible.

If A is boolean, then each $\wedge$-irreducible is maximal.

If A is linear, then each non-$\top$ element is $\wedge$-irreducible.

Proof. Suppose p is maximal in A. Thus $p < \top$ and there is nothing strictly between these two. Consider $x, y \in A$ with $x \wedge y \leq p$. If $x \not\leq p$ then $p < p \vee x$, and hence $p \vee x = \top$. Similarly, if $y \not\leq p$ then $p \vee y = \top$. Thus, if $x \not\leq p$ and $y \not\leq p$, then

$$p = p \vee (x \wedge y) = (p \vee x) \wedge (p \vee y) = \top$$

which is not so.

Suppose A is boolean and p is $\wedge$-irreducible in A. To show that p is maximal, consider any element x with $p < x$. With $y = \neg x$ we have $x \wedge y = \bot \leq p$, so that $y \leq p$ (since $x \not\leq p$). But now $y \leq p \leq x$ so that $\top = x \vee y \leq x$ to give the required result.

Suppose A is linear. Thus

$$x \leq y \quad \text{or} \quad y \leq x$$

for each $x, y \in A$. In particular we have

$$x \wedge y = x \quad \text{or} \quad x \wedge y = y$$

depending on how x and y compare. Now consider any element $p < \top$. For each $x, y \in A$ we have

$$x \wedge y \leq p \Longrightarrow x = x \wedge y \leq p \quad \text{or} \quad y = x \wedge y \leq p$$

to show that p is $\wedge$-irreducible. ■

The second part of this result shows that there are some quite large frames that have no points at all. The points of a complete boolean algebra are its maximal elements and these are in bijective correspondence with its minimal elements, usually called its atoms.

Thus a complete atomless boolean algebra is an example of a frame with no points. There are also some quite exotic, non-boolean, frames that have no points. We look at one of these in Section 6.

Definition 1.2 attaches to each frame A a set $\mathsf{pt}(A)$ of points. For each space S the topology $\mathcal{O}S$ is a frame. So what is the set of points $\mathsf{pt}(\mathcal{O}S)$? The obvious answer is not quite correct.

Let S be an arbitrary space. Observe that for each $s \in S$ and $U \in \mathcal{O}S$ we have

$$U \subseteq s^{-\prime} \iff s^{-} \subseteq U' \iff s \in U' \iff s \notin U$$

and, of course, $s^{-\prime}$ is also a member of $\mathcal{O}S$.

1.3 LEMMA. *For each space S, each point $s \in S$ gives a point $s^{-\prime}$ of $\mathcal{O}S$.*

Proof. We have $s^{-\prime} \neq S$ (the top of $\mathcal{O}S$), for otherwise $s^{-} = \emptyset$, which is not so.

For each $U, V \in \mathcal{O}S$ the observation above gives

$$U \cap V \subseteq s^{-\prime} \Longrightarrow s \notin U \cap V \Longrightarrow s \notin U \text{ or } s \notin V \Longrightarrow U \subseteq s^{-\prime} \text{ or } V \subseteq s^{-\prime}$$

to show that $s^{-\prime}$ is $\cap$-irreducible in $\mathcal{O}S$. ■

This result sets up an assignment

$$\begin{array}{rcl} S & \longrightarrow & \mathsf{pt}(\mathcal{O}S) \\ s & \longmapsto & s^{-\prime} \end{array}$$

which, in due course, we see is a continuous map. However, it need not be injective, nor surjective. We look at the ramifications of this in Section 3.

The important thing is that when dealing with a space S we should remember to distinguish between a point of S and a point of $\mathcal{O}S$.

We now return to the general construction of furnishing $\mathsf{pt}(A)$ with a topology.

1.4 DEFINITION. Let A be a frame. For each element $a \in A$ we use

$$p \in U_A(a) \iff a \nleq p$$

(for $p \in \mathsf{pt}(A)$) to extract a subset of $\mathsf{pt}(A)$. ■

The subscript on U_A indicates the parent frame, but we often omit this when there is little chance of confusion. Here is an example where we do omit the subscript.

1.5 LEMMA. *For each frame A we have*

$$\begin{array}{ll} U(\top) = \mathsf{pt}(A) & U(\bot) = \emptyset \\ U(a \wedge b) = U(a) \cap U(b) & U(\bigvee X) = \bigcup U^{\rightarrow}(X) \end{array}$$

for each $a, b \in A$ and $X \subseteq A$.

Proof. Of these only the bottom left equality is not immediate. This depends on the $\wedge$-irreducibility of the points in $\mathsf{pt}(A)$. Thus, for $p \in \mathsf{pt}(A)$ we have

$$p \notin U(a \wedge b) \iff a \wedge b \leq p \iff a \leq p \text{ or } b \leq p \iff p \notin U(a) \text{ or } p \notin U(b)$$

which, after taking the contrapositive, gives the equality. ■

This result shows that the range $U_A^{\rightarrow}(A)$ of the assignment U_A is a topology on $\mathsf{pt}(A)$, and so justifies the following definition.

1.6 DEFINITION. For each frame A the set $\mathsf{pt}(A)$ furnished with the topology

$$\mathcal{O}\mathsf{pt}(A) = \{U_A(a) \mid a \in A\}$$

is the point space of A. ■

Lemma 1.5 also gives the following.

1.7 COROLLARY. *For each frame A the assignment*

$$A \xrightarrow{U_A} \mathcal{O}\mathsf{pt}(A)$$

is a surjective frame morphism.

The following terminology will be justified in Section 3.

1.8 DEFINITION. For each frame A the surjective frame morphism U_A is the spatial reflection of A. ■

Each surjective frame morphism from a frame is determined by its kernel, the corresponding nucleus on that frame. For the spatial reflection this is easy to describe.

1.9 LEMMA. *For each frame A the kernel s_A of the spatial reflection U_A is given*

$$s_A(a) = \bigwedge\{p \in \mathsf{pt}(A) \mid a \leq p\}$$

(for each $a \in A$), where this infimum is computed in A.

Proof. For each $a \in A$ let

$$P(a) = \{p \in \mathsf{pt}(A) \mid a \leq p\}$$

so that $s(a) = \bigwedge P(a)$ is required. For each $x \in A$ we have

$$\begin{aligned} x \leq s(a) &\iff U(x) \subseteq U(a) \\ &\iff (\forall p \in \mathsf{pt}(A))[x \not\leq p \Longrightarrow a \not\leq p] \\ &\iff (\forall p \in \mathsf{pt}(A))[a \leq p \Longrightarrow x \leq p] \quad \iff x \leq \bigwedge P(a) \end{aligned}$$

to give the required result. ■

Not every space can arise as the point space of some frame. In Section 3 we characterize the spaces which do arise in this way. (They are the sober spaces.) For now we determine just some of their properties.

1.10 LEMMA. *For each frame A the specialization order on $\mathsf{pt}(A)$ is the reverse of the comparison inherited from A. In particular, $\mathsf{pt}(A)$ is T_0.*

Proof. For $p, q \in \mathsf{pt}(A)$ we have

$$\begin{aligned} q \sqsubseteq p &\iff q^- \subseteq p^- \\ &\iff (\forall x \in A)[q \in U(x) \implies p \in U(x)] \\ &\iff (\forall x \in A)[x \le p \implies x \le q] \qquad \iff p \le q \end{aligned}$$

to give the required result. ■

The assignment

$$A \longmapsto \mathsf{pt}(A)$$

is the object part of a contravariant functor, but what is the arrow part? How can we convert a frame morphism

$$B \xrightarrow{\ h\ } A$$

into a continuous map

$$\mathsf{pt}(A) \longrightarrow \mathsf{pt}(B)$$

between the point spaces? Of course, this conversion must interact with function composition in an appropriate manner.

Once again we remember that each frame morphism has a right adjoint

$$B \underset{h_*}{\overset{h^*}{\rightleftarrows}} A$$

passing in the other direction.

1.11 LEMMA. *For each frame morphism, as above, we have $h_*(p) \in \mathsf{pt}(B)$ for each $p \in \mathsf{pt}(A)$.*

Proof. Consider $p \in \mathsf{pt}(A)$. For each $z \in B$ we have

$$z \le h_*(p) \iff h^*(z) \le p$$

and hence $h_*(p) \ne \top$. For each $x, y \in B$ we have

$$\begin{aligned} x \wedge y \le h_*(p) &\implies h^*(x) \wedge h^*(y) = h^*(x \wedge y) \le p \\ &\implies h^*(x) \le p \text{ or } h^*(y) \le p \qquad \implies x \le h_*(p) \text{ or } y \le h_*(p) \end{aligned}$$

to show that $h_*(p)$ is $\wedge$-irreducible. ■

This gives us both an object assignment and an arrow assignment from ***Frm*** to ***Top***. For the time being let us be a bit pedantic, For each frame morphism h, as above, let

$$h_\star = h_*|_{\mathsf{pt}(A)}$$

so that we have a function

$$h_\star : \mathsf{pt}(A) \longrightarrow \mathsf{pt}(B)$$

be Lemma 1.11. (There may be time when we get a bit sloppy and write h_* for $h_\star$.)

1.12 THEOREM. *The pair of assignments*

$$A \longmapsto \mathsf{pt}(A) \qquad\qquad h \longmapsto h_\star$$

form a contravariant functor

$$\boldsymbol{Frm} \longrightarrow \boldsymbol{Top}$$

from frames to spaces.

Proof. There is very little left to check, but we should show that for each frame morphism

$$B \xrightarrow{\ h\ } A$$

the induced function

$$h_\star : \mathsf{pt}(A) \longrightarrow \mathsf{pt}(B)$$

is continuous. To do that we show

$$h_\star^{\leftarrow}(U_B(b)) = U_A(h(b))$$

for each $b \in B$.

For each point $p \in \mathsf{pt}(A)$ we have

$$\begin{aligned} p \in h_\star^{\leftarrow}(U_B(b)) &\Longleftrightarrow h_\star(p) \in U_B(b) \\ &\Longleftrightarrow b \not\leq h_*(p) \\ &\Longleftrightarrow h^*(b) \not\leq p \qquad \Longleftrightarrow p \in U(A(h(b)) \end{aligned}$$

to give the required result. The adjunction $h = h^* \dashv h_*$ is used at the third equivalence.

Let's also look at the passage across composition of arrows.

Consider a composite

$$C \xrightarrow{\ k\ } B \xrightarrow{\ h\ } A$$

of frame morphisms. Each component has a right adjoint

$$C \xleftarrow[\ k_*\ ]{} B \xleftarrow[\ h_*\ ]{} A$$

and we find that

$$(h \circ k)_* = k_* \circ h_*$$

which leads to the required property. ■

The calculation in this proof shows that each frame morphism

$$B \xrightarrow{\ h\ } A$$

induces a commuting square

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "U_B"'] & A \arrow[d, "U_A"] \\ \mathcal{O}\mathsf{pt}(B) \arrow[r, "h_\star^{\leftarrow}"'] & \mathcal{O}\mathsf{pt}(A) \end{tikzcd}$$

of frame morphisms. Thus we have the following.

1.13 SCHOLIUM. *The composite $\mathcal{O} \circ \mathsf{pt}$ is an endofunctor on* $\boldsymbol{Frm}$, *and the indexing morphism $U_\bullet$ is natural.*

On the whole there it little point in setting up a functor unless some use can be made of it. Here we have set up a pair a functors. Why?

1.14 THEOREM. *For each frame A and space S there is a bijective correspondence*

$$\begin{array}{ccc} \boldsymbol{Frm}[A, \mathcal{O}S] & & \boldsymbol{Top}[S, \mathsf{pt}(A)] \\ f & \longleftrightarrow & \phi \end{array}$$

between frame morphisms and space maps given by

$$s \in f(a) \Longleftrightarrow a \nleq \phi(s)$$

(for $a \in A$ and $s \in S$). Furthermore, this correspondence is natural for variations of A and S.

Proof. Suppose that f is a frame morphism (with indicated source and target). We need to do some checks on the suggested ϕ.

The first thing we need to check is that such a function ϕ does exists. To do this fix $s \in S$, let $X \subseteq A$ be given by

$$x \in X \Longleftrightarrow s \notin f(x)$$

(for $x \in A$), and let $\phi(s) = \bigvee X$. Since f is a frame morphism we have

$$f(\phi(s)) = \bigcup \{f(x) \mid x \in X\}$$

and hence $s \notin f(\phi(s))$. Thus, for $a \in A$ we have

$$a \leq \phi(s) \Longrightarrow f(a) \subseteq f(\phi(s)) \Longrightarrow s \notin f(a)$$

and the converse

$$s \notin f(a) \Longrightarrow a \in X \Longrightarrow a \leq \phi(s)$$

is little more than the definition of $\phi(s)$.

This gives a function

$$\phi : S \longrightarrow A$$

so it remains to check that each value of ϕ is a point of A and that ϕ is continuous.

Consider $s \in S$. We have $\phi(s) \neq \top$, otherwise $s \notin f(\top)$.

For $s \in S$ the element $\phi(s)$ of A is $\wedge$-irreducible. For otherwise we have some $a, b \in A$ with

$$a \nleq p(s) \qquad b \nleq \phi(s) \qquad a \wedge b \leq \phi(s)$$

and then

$$s \in f(a) \qquad s \in f(b) \qquad s \notin f(a \wedge b)$$

which is a contradiction since $f(a \wedge b) = f(a) \cap f(b)$.

The map ϕ is continuous. Each open set of $\mathsf{pt}(A)$ has the form $U(a)$ for some $a \in A$, and

$$\phi^{\leftarrow}(U(a)) = f(a)$$

by a simple calculation. This deals with the assignment $f \longmapsto \phi$.

Suppose that ϕ is a space map (with indicated source and target). We need to do some checks on the suggested f.

For each $a \in A$ a simple calculation gives

$$f(a) = \phi^{\leftarrow}(U(a))$$

to show that $f(a) \in \mathcal{O}S$. Furthermore, since both

$$A \xrightarrow{U_A} \mathcal{O}S \xrightarrow{\phi^{\leftarrow}} \mathcal{O}\mathsf{pt}(A)$$

are frame morphisms, the composite

$$f = \phi^{\leftarrow} \circ U_A$$

is a frame morphism. This deals with the assignment $\phi \longmapsto f$.

Consider the two 2-steps trips

$$f \longmapsto \phi \longmapsto g \qquad \phi \longmapsto f \longmapsto \psi$$

across the assignments. For each $a \in A$ and $s \in S$ we have

$$s \in g(a) \Longleftrightarrow a \not\leq \phi(a) \Longleftrightarrow s \in f(a) \qquad a \leq \psi(s) \Longleftrightarrow s \notin \phi(a) \Longleftrightarrow a \leq \phi(s)$$

to show

$$g(a) = f(a) \qquad \psi(s) = \phi(s)$$

and so verify the bijective correspondence.

It remains to deal with the naturality. To this end consider a pair or arrows

$$B \xrightarrow{h} A \qquad T \xrightarrow{\theta} S$$

one from $\boldsymbol{Frm}$ and one from $\boldsymbol{Top}$. These induce a square of assignments

$$\begin{tikzcd}
f \arrow[rr, leftrightarrow] & & \phi \\
\boldsymbol{Frm}[A, \mathcal{O}S] \arrow[d, "\theta^{\rightarrow} \circ - \circ h"'] & & \boldsymbol{Top}[S, \mathsf{pt}(A)] \arrow[d, "h_* \circ - \circ \theta"] \\
\boldsymbol{Frm}[B, \mathcal{O}T] & & \boldsymbol{Top}[T, \mathsf{pt}(B)] \\
g \arrow[rr, leftrightarrow] & & \psi
\end{tikzcd}$$

where the vertical assignments are as indicated. We must show that the two paths from top left to bottom right agree, and the two paths from top right to bottom left agree. (Actually, it is sufficient to deal with just one of these pairs.) But for $b \in B$ and $t \in T$ we have

$$t \in (\theta^{\leftarrow} \circ f \circ h)(b) \Longleftrightarrow \theta(t) \in f(h(b)) \qquad b \leq (h_* \circ \phi \circ \theta)(t) \Longleftrightarrow h(b) \leq \phi(\theta(t))$$

so that given correspondence

$$f \longleftrightarrow \phi$$

ensures the required results. ■

This result shows that the two functors $\mathcal{O}(\cdot)$ and $\mathsf{pt}(\cdot)$ form a contravariant adjunction between the categories ***Frm*** and ***Top***. Such adjunctions have several different characterizations and properties. We look at some of these for this particular adjunction in the next sections.

To conclude this section we remember that each contravariant has an associated pair of units, each obtained by transposing an identity arrow to the other side. What do you think the units of this particular adjunctions are? Your guess won't be far wrong.

For a frame A or space S the unit is obtained as follows.

$$\begin{array}{ccccccc} \boldsymbol{Frm}[A, \mathcal{O}\mathsf{pt}(A)] & & \boldsymbol{Top}[\mathsf{pt}(A), \mathsf{pt}(A)] & \qquad & \boldsymbol{Frm}[\mathcal{O}S, \mathcal{O}S] & & \boldsymbol{Top}[S, \mathsf{pt}(\mathcal{O}S)] \\ ? & \longleftarrow\!\!\!\shortmid & id_{\mathsf{pt}(A)} & & id_{\mathcal{O}S} & \longmapsto & ? \end{array}$$

We easily determine what these are.

1.15 THEOREM. *(f) For each frame A the unit*

$$A \longrightarrow \mathcal{O}\mathsf{pt}(A)$$

is the spatial reflection U_A. In particular, this unit U_A is a frame morphism.

(s) For each space S the unit

$$\begin{array}{ccc} S & \longrightarrow & \mathsf{pt}(\mathcal{O}S) \\ s & \longmapsto & s^{-\prime} \end{array}$$

send a point of S to the corresponding point of $\mathcal{O}S$. In particular, this unit is a continuous map.

Proof. (f) Let ϕ be the identity map on $\mathsf{pt}(A)$, and let f be the transpose of ϕ as given by Theorem 1.14. For each $a \in A$ and $s \in \mathsf{pt}(A)$ we have

$$s \in f(a) \iff a \nleq \phi(s) \iff a \nleq s \iff s \in U_A(a)$$

to show that $f = U_A$. Of course, we checked that U_A is a frame morphism before we set up the topology on $\mathsf{pt}(A)$.

(s) Let f be the identity morphism of $\mathcal{O}S$, and let ϕ be the transpose of f as given by Theorem 1.14. Remembering the equivalence just before Lemma 1.3, for each $s \in S$ and $U \in \mathcal{O}S$ we have

$$U \subseteq \phi(s) \iff s \notin f(U) \iff s \notin U \iff U \subseteq s^{-\prime}$$

to show that $\phi(s) = s^{-\prime}$. The continuity of ϕ can be checked directly. ■

This sets up the contravariant adjunction in a rather computational fashion. As I said in the Preamble, the construction does look at bit *ad hoc*, and that is because we have not yet been told the full story. That is the topic of the next section.

2 The schizophrenic adjunction

As I said in the preamble to this document, the construction of Section 1 can seem rather *ad hoc*, and prompts us to ask questions. Why that particular construction, and not some other? What benefits can we gain from the construction? Is there some secret not being told?

In this section we learn that the construction is an instance of a more general, and quite canonical, construction. It is a schizophrenically induced contravariant adjunction. These arise in several places in mathematics, especially when there is a duality around. By now the technique used ought to be set in stone.

We also see that the adjunction could be set up in a different way, in fact several different ways. In isolation each of these looks *ad hoc*, but when seen as a whole we begin to appreciate that there is something more fundamental going on.

It turns out that the version of the construction describe in Section 1 is the one that is most often useful in practice. Nevertheless, sometimes other versions are useful, and these are uncovered in this section.

We want to connect the two categories

$$\boldsymbol{Frm} \qquad \boldsymbol{Top}$$

as in Theorem 1.14. To do this we find a single gadget which can pose as an object of either category. The particular gadget we use is

$$2 = \{0, 1\}$$

the 2-element set.

This can pose as a frame with $0 < 1$. In fact, this is the initial frame.

It can also pose as a space with

$$\mathcal{O}2 = \{\emptyset, \{1\}, 2\}$$

as the carried topology. This is usually called sierpinski space.

The trick is to remember that for each set Z there is a bijective correspondence

$$\begin{array}{ccc} \mathcal{P}Z & & [S \longrightarrow 2] \\ E & \longleftrightarrow & \chi \end{array}$$

between subsets E of Z and the characters (characteristic functions) χ on Z. Here it is convenient to write $E^\wedge$ for the character corresponding to the subset E. Thus

$$z \in E \iff E^\wedge(z) = 1$$
$$z \notin E \iff E^\wedge(z) = 0$$

for $z \in Z$. To fit in with later terminology we refer to such a characteristic function as a ***Set***-character.

We make a trivial, but significant, observation.

2.1 LEMMA. *Let S be a space. A subset of S is open precisely when its character is continuous (relative to the given topology on S and the sierpinski topology on 2).*

Proof. Consider any character

$$\chi : S \longrightarrow 2$$

on S. This is continuous precisely when each of

$$\chi^{\leftarrow}(\emptyset) \quad \chi^{\leftarrow}(\{1\}) \quad \chi^{\leftarrow}(2)$$

is open in S. The two outside ones are $\emptyset$ and S, and these are always open. Let

$$E = \chi^{\leftarrow}(\{1\})$$

so that $\chi = E^{\wedge}$. We have just see that

$$\chi \text{ is continuous} \Longleftrightarrow E \text{ is open}$$

which is what we have to prove. ■

This result shows that for each space S we have a bijective correspondence

$$\begin{array}{ccc} \mathcal{O}S & & \boldsymbol{Top}[S,2] \\ U & \longleftrightarrow & U^{\wedge} \end{array}$$

between the carried topology and the indicated arrow set of the category $\boldsymbol{Top}$. We call $\boldsymbol{Top}[S,2]$ the set of $\boldsymbol{Top}$-characters, or the set of space characters of S, or even just the characters of S when it is clear we are dealing with topological aspects.

We can take this a bit further.

Consider a continuous map

$$T \xrightarrow{\ \theta\ } S$$

between spaces. This induces a frame morphism $\theta^{\leftarrow}$ between the topologies. In turn this induces a function θ° between the two sets of space characters.

$$\begin{array}{ccccc} & & U & \longleftrightarrow & U^{\wedge} \\ S & & \mathcal{O}S & & \boldsymbol{Top}[S,2] \\ \uparrow{\scriptstyle\theta} & & {\scriptstyle\theta^{\leftarrow}}\downarrow & & \downarrow{\scriptstyle\theta^{\circ}} \\ T & & \mathcal{O}T & & \boldsymbol{Top}[T,2] \\ & & V & \longleftrightarrow & V^{\wedge} \end{array}$$

For each $U \in \mathcal{O}S$ we have

$$\theta^{\circ}(U^{\wedge}) = \theta^{\leftarrow}(U)^{\wedge}$$

so that

$$\begin{aligned} \theta^{\circ}(U^{\wedge})(t) = 1 &\Longleftrightarrow \theta^{\leftarrow}(U)^{\wedge}(t) = 1 \\ &\Longleftrightarrow t \in \theta^{\leftarrow}(U) \\ &\Longleftrightarrow \theta(t) \in U \qquad \Longleftrightarrow U^{\wedge}(\theta(t)) = 1 \end{aligned}$$

for each $t \in T$. In other words

$$\theta^{\circ}(U^{\wedge}) = U^{\wedge} \circ \theta$$

for each $U \in \mathcal{O}S$.

These calculations almost prove the following.

2.2 LEMMA. *The contravariant functor*

$$\textbf{\textit{Top}} \xrightarrow{\mathcal{O}} \textbf{\textit{Frm}}$$

is naturally isomorphic to the 'enriched' hom-functor $\textbf{\textit{Top}}[-, \mathsf{2}]$.

We won't go into the precise meaning of this result (for it that doesn't help us to understand it). However, a bit more explanation won't go amiss.

Notice that the calculations above show that the two *sets*

$$\mathcal{O}S \qquad \textbf{\textit{Top}}[S, \mathsf{2}]$$

are naturally isomorphic for variation of the space S. What we haven't done is furnish $\textbf{\textit{Top}}[S, \mathsf{2}]$ with a frame structure and show that this matches that of $\mathcal{O}S$. This is not hard to do, but there is no great benefit in doing it here. (The required structure is obtained by a pointwise lifting from that on $\mathsf{2}$). The important message is that the behaviour of the functor $\mathcal{O}(\cdot)$ is that of a hom-functor with a few extra twiddly bits.

We are now going to obtain a similar description of the functor $\mathsf{pt}(\cdot)$ in the opposite direction.

Let A be a frame. A sf $\textbf{\textit{Frm}}$-character or a frame character of A is a member of the arrow set $\textbf{\textit{Frm}}[A, \mathsf{2}]$, that is a frame morphism

$$A \xrightarrow{\chi} \mathsf{2}$$

from A to the 2-element frame $\mathsf{2}$. Thus

$$\begin{array}{cc} \chi(\top) = 1 & \chi(\bot) = 0 \\ \chi(x \wedge y) = \chi(x) \wedge \chi(y) & \chi(\bigvee X) = \bigvee \chi^{\rightarrow}(X) \end{array}$$

for each $x, y \in A$ and $X \subseteq A$. You might like to ponder on the bottom right condition, and show that these characters are in bijective correspondence with the $\wedge$-irreducible elements of A. (We look at the details of this proof shortly.)

We also use a different kind of gadget. Recall that a filter on the frame A is a non-empty upper section F of A such that

$$x, y \in F \Longrightarrow x \wedge y \in F$$

for $x, y \in A$. A filter F is proper if $\bot \notin F$. A filter F is prime if it is proper and

$$x \vee y \in F \Longrightarrow x \in F \text{ or } y \in F$$

(for $x, y \in A$). We restrict this notion even further.

2.3 DEFINITION. Let A be a frame. A filter P on A is completely prime if it is proper and

$$\bigvee X \in P \Longrightarrow X \cap P \neq \emptyset$$

for each $X \subseteq A$. ■

By considering a pair $X = \{x, y\}$ we see that each completely prime filter is prime. However, a prime filter need not be completely prime.

Here is why these filters are useful.

2.4 LEMMA. *For each frame A there are bijective correspondences between the following gadgets on A.*

($\uparrow$) *Completely prime filters P on A.*

($\leftrightarrow$) *Frame characters χ on A.*

($\downarrow$) *Elements p of A that are $\wedge$-irreducible.*

These correspondences are given by

$$x \le p \Longleftrightarrow \chi(x) = 0 \qquad \chi(x) = 1 \Longleftrightarrow x \in P$$

for $x \in A$.

Proof. The proof consists of a series of small observations.

Consider a completely prime filter P on A, and let χ be the suggested function. We have

$$\chi(x) = 1 \Longleftrightarrow x \in P$$
$$\chi(x) = 0 \Longleftrightarrow x \notin P$$

for $x \in A$. In other words χ is the characteristic function of P as a subset of A. We must show that χ is a frame character, that is we must show the four conditions listed above.

Remembering that χ can take only two values, these conditions translate into

$$\begin{array}{cc} \top \in P & \bot \notin P \\ x \wedge y \in P \iff x \in P \text{ and } y \in P & \bigvee X \in P \iff X \cap P \neq \emptyset \end{array}$$

for $x, y \in A$ and $X \subseteq A$. The various properties of P ensure that these conditions hold.

Consider a frame character χ on A, and let P be the suggested subset. Thus χ and P are related by the two equivalences give in the previous block. We must show that P is a completely prime filter on A.

Since $\chi(\top) = 1$, we have $\top \in P$, and hence P is non-empty.

Consider $x, y \in A$ with $x \le y$ and $x \in P$. Then

$$1 = \chi(x) \le \chi(y)$$

to give $\chi(y) = 1$, and hence $y \in P$. This shows that P is an upper section of A.

Consider $x, y \in P$. Then

$$\chi(x \wedge y) = \chi(x) \wedge \chi(y) = 1 \wedge 1 = 1$$

and hence $x \wedge y \in P$. This shows that P is a filter on A.

Since $\chi(\bot) = 0$ we have $\bot \notin P$, and hence P is proper.

Finally, consider $X \subseteq A$ with $\bigvee X \in P$. By way of contradiction suppose $X \cap P = \emptyset$. Thus $\chi(x) = 0$ for each $x \in X$, so that

$$\chi(\bigvee X) = \bigvee \chi^{\rightarrow}(X) = \bigvee \{0\} = 0$$

and hence $\bigvee X \notin P$, which is the contradiction.

From the constructions involved it is immediate that this sets up a bijective correspondence between completely prime filters and frame characters on A.

Consider a $\wedge$-irreducible element p on A, and let χ be the suggested function. We have

$$\chi(x) = 1 \iff x \not\leq p$$
$$\chi(x) = 0 \iff x \leq p$$

for $x \in A$. We must show that χ is a frame character, that is we must show the four conditions listed above.

Remembering that χ can take only two values, these conditions translate into

$$\begin{array}{cc} \top \not\leq p & \bot \leq p \\ x \wedge y \leq p \iff x \leq p \text{ or } y \leq p & \bigvee X \leq p \iff (\forall x \in X)[x \leq p] \end{array}$$

for $x, y \in A$ and $X \subseteq A$. The right hand properties are trivial, and the $\wedge$-irreducibility of p ensure the left hand properties.

Consider a frame character χ on A. Let

$$X = \{x \in A \mid \chi(x) = 0\}$$

so that

$$\chi(\bigvee X) = \bigvee \chi^{\rightarrow}(X) = \bigvee\{0\} = 0$$

and hence the supremum

$$p = \bigvee X$$

belongs to X. Thus

$$x \leq p \iff \chi(x) = 0$$

where the implication $\Rightarrow$ holds be the previous argument. We must show that p is $\wedge$-irreducible.

Since $\chi(\top) = 1$ we have $\top \not\leq p$, that is $p \neq \top$.

Consider $x, y \in A$ with $x \wedge y \leq p$. Then

$$\chi(x) \wedge \chi(y) = \chi(x \wedge y) = 0$$

so that

$$\chi(x) = 0 \text{ or } \chi(y) = 0$$

and hence

$$x \leq p \text{ or } y \leq p$$

to give the required result.

Using these calculations we see that this sets up a bijective correspondence between the $\wedge$-irreducible elements of A and frame the characters on A. ■

Amongst other things Lemma 2.4 shows that for each frame A we have a bijective correspondence

$$\begin{array}{ccc} \mathsf{pt}(A) & & \boldsymbol{Frm}[A, 2] \\ p & \longleftrightarrow & p^{\wedge} \end{array}$$

between the point space of A (as the set of $\wedge$-irreducible elements) and the set of frame characters on A. For convenience we use a similar notation as with the space correspondence, Thus

$$x \le p \iff p^\wedge(x) = 0$$

for each $x \in A$.

As with the space case, we can take this a bit further.

Consider a morphism

$$B \xrightarrow{\ h\ } A$$

between frames. This induces a continuous map $h_\star$ (h_* restricted to $\mathsf{pt}(A)$) between the spaces. In turn this induces a function h° between the two sets of frame characters.

$$\begin{tikzcd} A & p \arrow[r, leftrightarrow] & p^\wedge \\ & \mathsf{pt}(A) \arrow[d, "h_\star"'] & \boldsymbol{Frm}[A, 2] \arrow[d, "h^\circ"] \\ B \arrow[uu, "h"'] & \mathsf{pt}(B) & \boldsymbol{Frm}[B, 2] \\ & q \arrow[r, leftrightarrow] & q^\wedge \end{tikzcd}$$

For each $p \in \mathsf{pt}(A)$ we have

$$h^\circ(p^\wedge) = h_*(p)^\wedge$$

so that

$$\begin{aligned} h^\circ(p^\wedge)(b) = 0 &\iff h_*(p)^\wedge(b) = 0 \\ &\iff b \le h_*(p) \\ &\iff h(b) \le p \qquad\qquad \iff p^\wedge(h(b)) = 0 \end{aligned}$$

to give

$$h^\circ(p^\wedge)(b) = p^\wedge(h(b))$$

for each $b \in B$. Thus

$$h^\circ(p^\wedge) = p^\wedge \circ h$$

for each $p \in \mathsf{pt}(A)$.

These calculations almost prove the following analogue of Lemma 2.2.

2.5 LEMMA. *The contravariant functor*

$$\boldsymbol{Frm} \xrightarrow{\ \mathsf{pt}\ } \boldsymbol{Top}$$

is naturally isomorphic to the 'enriched' hom-functor $\boldsymbol{Frm}[-, 2]$.

As with Lemma 2.2 we won't go into the precise meaning of this result. However, you might like to check that $\boldsymbol{Frm}[A, 2]$ is a subspace of $\boldsymbol{Set}[A, 2]$ furnished with the product topology lifted from the sierpinski topology on 2.

These bijective correspondences can be used to explain Theorem 1.14. For each frame A and space S we have several correspondences.

$$\begin{tikzcd} f \arrow[rr, leftrightarrow] & & \phi \\[-2em] \boldsymbol{Frm}[A, \mathcal{O}S] \arrow[d, leftrightarrow] & & \boldsymbol{Top}[S, \mathsf{pt}(A)] \arrow[d, leftrightarrow] \\ \boldsymbol{Frm}[A, \boldsymbol{Top}[S, 2]] & & \boldsymbol{Top}[S, \boldsymbol{Frm}[A, 2]] \\[-2em] f^\wedge \arrow[rr, leftrightarrow] & & \phi^\wedge \end{tikzcd}$$

Across the top we have the correspondence of Theorem 1.14. Down either side we have the correspondences induced by the character correspondences

$$\mathcal{O}S \longleftrightarrow \boldsymbol{Top}[S, 2] \qquad \mathsf{pt}(A) \longleftrightarrow \boldsymbol{Frm}[A, 2]$$

respectively. These give

$$f^\wedge(a) = f(a)^\wedge \qquad \phi(s)^\wedge = \phi^\wedge(s)$$

and in particular

$$f^\wedge(a)(s) = 1 \iff s \in f(a) \qquad a \leq \phi(s) \iff \phi^\wedge(s)(a) = 0$$

for each $a \in A$ and $s \in S$. Combining these three gives the correspondence across the bottom. Thus we have

$$f^\wedge(a)(s) = \phi^\wedge(s)(a)$$

for each $a \in A$ and $s \in S$.

These calculations indicate that the contravariant adjunction produced in Section 1 is not the result of a *ad hoc* construction. At the set level it is nothing more than

$$[A \longrightarrow [S \longrightarrow 2]] \cong [A \times S \longrightarrow 2] \cong [S \times A \longrightarrow 2] \cong [S \longrightarrow [A \longrightarrow 2]]$$

where we curry the functions on the outside and chip the inputs on the inside. However, as we have seen, this is not the whole story, we have to furnish $\mathsf{pt}(A)$ and $\mathcal{O}S$ in the appropriate way, and make sure the correspondences respect the furnishings.

The most important result in this section is Lemma 2.4. In Section 1 we constructed $\mathsf{pt}(A)$ using the $\wedge$-irreducible elements p of A as the points. We could equally well use the frame characters χ of A, or the completely prime filters P on A. In these terms the indexing morphism

$$A \xrightarrow{U_A} \mathsf{pt}(A)$$

is given by

$$\begin{array}{cl} (\uparrow) & P \in U(a) \iff a \in P \\ (\leftrightarrow) & \chi \in U(a) \iff \chi(a) = 1 \\ (\downarrow) & p \in U(a) \iff a \nleq p \end{array}$$

for each $a \in A$. In particular, $(\uparrow)$ is just a standard filter space construction. Each of these versions is useful at various times, but it is the $\wedge$-irreducible version that we use most often.

3 Reflections on sobriety

Theorem 1.14 sets up a contravariant adjunction between the two categories ***Frm*** and ***Top***. As usual, any such adjunction can be described in several different ways, and often such rephrasings brings out some hidden features.

Each frame A has a canonical morphism

$$A \xrightarrow{U_A} \mathcal{O}\mathsf{pt}(A)$$

to the topology of its point space. We call this the spatial reflection of A. When it was first introduced, in Definition 1.8, this terminology was not justified. A reformulation of Theorem 1.14 provides the missing justification.

3.1 THEOREM. *Let A be a frame. For each space S and frame morphism*

$$A \xrightarrow{f} \mathcal{O}S$$

there is a unique continuous map

$$S \xrightarrow{\phi} \mathsf{pt}(A)$$

such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "U_A"'] & & \mathcal{O}S \\ & \mathcal{O}\mathsf{pt}(A) \arrow[ur, "\phi^{\leftarrow}"'] & \end{tikzcd}$$

commutes.

Proof. We are looking for a continuous map ϕ such that

$$\phi^{\leftarrow} \circ U_a = f$$

holds. In other words, we require

$$s \in f(a) \Longleftrightarrow s \in \phi^{\leftarrow}(U(a)) \Longleftrightarrow \phi(s) \in U(a) \Longleftrightarrow a \nleq \phi(s)$$

for all $a \in A$ and $s \in S$. Thus ϕ is nothing more than the transpose of f across the adjunction of Theorem 1.14. ■

Each contravariant adjunction has a pair of unit arrows obtained by transposing the appropriate identity arrow across the adjunction. By Theorem 1.14 we know that the frame unit is just the spatial reflection morphism, and the continuous map

$$\begin{array}{ccc} S & \longrightarrow & \mathsf{pt}(\mathcal{O}S) \\ s & \longmapsto & s^{-\prime} \end{array}$$

is the space unit. Is this a reflection of any kind? In this section we answer that question.

What is the target space $\mathsf{pt}(\mathcal{O}S)$ of this space unit? The obvious answer is that it is a homeomorphic copy of the source space S. This is incorrect for two reasons.

By Lemma 1.10 we know that each point space $\mathsf{pt}(A)$ is T_0. Thus if S is not T_0 then it can not be homeomorphic to $\mathsf{pt}(\mathcal{O}S)$.

3.2 LEMMA. *For a space S the space unit of S is injective precisely when S is T_0.*

Proof. A simple argument shows that a space S is T_0 precisely when

$$s^- = t^- \Longrightarrow s = t$$

for $s, t \in S$. ■

This may look like a bit of nit-picking, for who wants to deal with non-T_0 spaces? However, even for a T_0 space the space unit need not be surjective, and that is a much more interesting story.

3.3 DEFINITION. A subset X of a space S is closed irreducible if it is closed, non-empty, and if

$$\left.\begin{array}{l} U \text{ meets } U \\ X \text{ meets } V \end{array}\right\} \Longrightarrow X \text{ meets } U \cap V$$

holds for all $U, V \in \mathcal{O}S$. ■

Observe that X is closed irreducible in S precisely when X' is a $\wedge$-irreducible member of the frame $\mathcal{O}S$. This connection lies at the heart of much of what follows.

Examples of closed irreducible subsets of a space S are easy to find. For each point $s \in S$ and open set $U \in \mathcal{O}S$ we have

$$s^- \text{ meets } X \Longleftrightarrow s \in U$$

and hence the point closure s^- is closed irreducible.

3.4 EXAMPLE. Let S be an infinite set, and let $\mathcal{O}S$ be the cofinite topology on $\mathcal{O}S$. Thus, apart from S the closed sets are precisely the finite subsets of S. In particular, each singleton is closed, so S is T_1.

Suppose $U, V \in \mathcal{O}S$ both meet S, that is both are non-empty. Then $U' \cup V'$ is finite, and hence

$$U \cap V = (U' \cup V')'$$

is non-empty, and so meets S. This shows that S is closed irreducible. However, S is not a point closure (since it is not a singleton). ■

We may regard the existence of a closed irreducible subset which is not a point closure as a defect of the parent space, and look for a corrective measure.

3.5 DEFINITION. A space S is sober if each closed irreducible subset X has a unique generic point, that is $X = s^-$ for a unique point s. ■

A few simple arguments show that

$$T_2 \Longrightarrow \text{Sober} \Longrightarrow T_0$$

and neither of these implications is reversible. It is the T_0 property that ensures that a closed irreducible set can have no more than one generic point. The two properties

$$T_1 \qquad \text{Sober}$$

are incomparable, a space can have one property without the other.

The following result extends part of Lemma 1.10.

3.6 THEOREM. *For each frame A the point space $\mathsf{pt}(A)$ is sober.*

Proof. By Lemma 1.10 the space $\mathsf{pt}(A)$ is T_0, so it suffices to show that each closed irreducible subset of $\mathsf{pt}(A)$ is a point closure.

We use the kernel s of the spatial reflection

$$A \xrightarrow{U} \mathcal{O}\mathsf{pt}(A)$$

of A. Thus

$$x \leq s(a) \iff U(x) \subseteq U(a)$$

for $x, a \in A$.

Suppose X is closed irreducible in $\mathsf{pt}(A)$. Since $X' \in \mathcal{O}\mathsf{pt}(A)$ we have

$$X' = U(a)$$

for at least one $a \in A$. Let $p = s(a)$, so that both

$$X' = U(p) \qquad s(p) = p$$

hold.

We show first that p is $\wedge$-irreducible in A.

We have $p \neq \top$, for otherwise $X' = U(\top) = S$, and hence X is empty.

We have

$$X \text{ meets } U(x) \iff U(x) \subseteq X' = U(p) \iff x \not\leq s(p) = p$$

for each $x \in A$. Thus, for $x, y \in A$, we have

$$\left.\begin{matrix} x \not\leq p \\ y \not\leq p \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} X \text{ meets } U(x) \\ X \text{ meets } U(y) \end{matrix}\right\} \Longrightarrow X \text{ meets } U(x) \cap U(y) = U(x \wedge y) \Longrightarrow x \wedge y \not\leq p$$

which, by taking the contrapositive, shows that p is $\wedge$-irreducible.

Secondly, we use specialization order of $\mathsf{pt}(A)$, as described in Lemma 1.10, to show $X = p^-$. For each $q \in \mathsf{pt}(A)$ we have

$$q \in X \iff q \notin U(p) \iff p \leq q \iff q \sqsubseteq p \iff q \in p^-$$

to give the required result. ■

This result shows that for a space S, if the canonical assignment

$$S \longrightarrow \mathsf{pt}(\mathcal{O}S)$$

is a homeomorphism then S must be sober. We will show the converse of this. More generally, we will show that the assignment is the sober reflection of S. To do that it is convenient to describe $\mathsf{pt}(\mathcal{O}S)$ is a slightly different way. In fact, we use the original construction of the sober reflection.

So far we have viewed the points of $\mathsf{pt}(\mathcal{O}S)$ as those open sets of S which are $\wedge$-irreducible in $\mathcal{O}S$ (with a nod towards the frame characters on $\mathcal{O}S$ and the completely prime filters on $\mathcal{O}S$). But an open set of S is $\wedge$-irreducible precisely when its complement is closed irreducible complement in S. We use these closed sets as the new points.

3.7 DEFINITION. For a space S let $\mathsf{sob}(S)$ be the family of closed irreducible subsets of S. For $U \in \mathcal{O}S$ let $\natural(U)$ be the subset of $\mathsf{sob}(S)$ given by

$$X \in \natural(U) \iff X \text{ meets } U$$

(for $X \in \mathsf{sob}(S)$). ■

We have seen this construction before, but in a slightly different guise. Some of the notation in the next paragraph is a bit hairy, but once we have got through it, we can forget it.

For a space S we have the indexing morphism

$$\mathcal{O}S \xrightarrow{U_{\mathcal{O}S}} \mathcal{O}\mathsf{pt}(\mathcal{O}S)$$

of the topology on $\mathsf{pt}(\mathcal{O}S)$. Temporarily, for $U \in \mathcal{O}S$ let $?(U)$ be the image of – and this is the hairy bit – the open set

$$U_{\mathcal{O}S}(U)$$

across the bijection to $\mathsf{sob}(S)$. Thus, for $X \in \mathsf{sob}(S)$ we have

$$X \in ?(U) \iff X' \in U_{\mathcal{O}S}(U) \iff U \nsubseteq X' \iff X \text{ meets } U$$

so that $?(U)$ is nothing more than $\natural(U)$.

You may check directly that

$$\mathcal{O}\mathsf{sob}(S) = \{\natural(U) \mid U \in \mathcal{O}S\}$$

is a topology on $\mathsf{sob}(S)$ with

$$\mathcal{O}S \xrightarrow{\natural} \mathcal{O}\mathsf{sob}(S)$$

as a surjective frame morphism. Furthermore, we have a homeomorphism

$$\mathsf{pt}(\mathcal{O}S) \longleftrightarrow \mathsf{sob}(S)$$

obtained by taking complements, and the assignment

$$\begin{array}{ccc} S & \xrightarrow{\varsigma} & \mathsf{sob}(S) \\ s & \longmapsto & s^- \end{array}$$

is continuous.

3.8 THEOREM. *Let S be a space.*

(a) The canonical frame morphism

$$\mathcal{O}S \xrightarrow{\natural} \mathcal{O}\mathsf{sob}(S)$$

is an isomorphism.

(b) The specialization order on $\mathsf{sob}(S)$ is given by inclusion (of the closed irreducible subsets of S). In particular, $\mathsf{sob}(S)$ is T_0.

(c) The space $\mathsf{sob}(S)$ is sober.

(d) The canonical continuous map

$$S \xrightarrow{\varsigma} \mathsf{sob}(S)$$

is injective precisely when S is T_0.

(e) This continuous map is a homeomorphism precisely when S is sober.

Proof. We need not give all the details but some of them are worth looking at.

(a) We stated above (and it is not hard to prove) that the frame morphism $\natural(\cdot)$ is surjective. Thus we need to check that it is injective. To do this we show

$$\natural(V) \subseteq \natural(U) \Longrightarrow V \subseteq U$$

for $U, V \in \mathcal{O}S$. To this end suppose $\natural(V) \subseteq \natural(U)$ and consider $s \in V$. Then $s^- \in \mathsf{sob}(S)$ and $s^- \in \natural(V)$ (since the two sets meet at s). Thus $s^- \in \natural(U)$, so that s^- meets U, to give some point t (of S) with $t \in s^- \cap U$. Remembering that each open in $\mathcal{O}S$ is an upper section in the specialization order of S, we have $s \in U$, to give the required result.

(b) To describe the specialization order of $\mathsf{sob}(S)$, consider $X, Y \in \mathsf{sob}(S)$. We have

$$Y \sqsubseteq X$$

precisely when X belongs to each open set of $\mathsf{sob}(S)$ to which Y belongs. Thus

$$\begin{aligned} Y \sqsubseteq X &\Longleftrightarrow (\forall U \in \mathcal{O}S)[Y \in \natural(U) \Longrightarrow X \in \natural(U)] \\ &\Longleftrightarrow (\forall U \in \mathcal{O}S)[Y \text{ meets } U \Longrightarrow X \text{ meets } U] \\ &\Longleftrightarrow (\forall U \in \mathcal{O}S)[X \cap U' = \emptyset \Longrightarrow Y \cap U' = \emptyset] \\ &\Longleftrightarrow (\forall Z \in \mathcal{C}S)[X \subseteq Z \Longrightarrow Y \subseteq Z] \qquad \Longleftrightarrow Y \subseteq X \end{aligned}$$

to give the required result.

This space $\mathsf{sob}(S)$ is T_0 since its specialization is a partial ordering.

(c) Since $\mathsf{sob}(S)$ is T_0, it suffices to show that each closed irreducible subset (of $\mathsf{sob}(S)$) has a generic point.

To this end consider any closed irreducible subset $\mathcal{Z}$ of $\mathsf{sob}(S)$. This has the form

$$\mathcal{Z} = \natural(Z')'$$

for some $Z \in \mathcal{C}S$. The tactic is to show first that Z is closed irreducible in S, so that $Z \in \mathsf{sob}(S)$, and then show that Z is a generic point of $\mathcal{Z}$.

We have

$$X \in \mathcal{Z} \Longleftrightarrow X \notin \natural(Z') \Longleftrightarrow X \cap Z' = \emptyset \Longleftrightarrow X \subseteq Z$$

that is

$$X \in \mathcal{Z} \Longleftrightarrow X \subseteq Z$$

for $X \in \mathsf{sob}(S)$. (Remember that, as yet, Z is just a closed subset of S whereas X is a closed irreducible subset.)

This equivalence ensures that

$$\mathcal{Z} \text{ meets } \natural(U) \Longleftrightarrow Z \text{ meets } U$$

for $U \in \mathcal{O}S$. If $\mathcal{Z}$ meets $\natural(U)$ then there is some $X \in \mathcal{Z} \cap \natural(U)$, that is $X \subseteq Z$ with $X \cap U \neq \emptyset$, and so Z meets U (at least in the same places that X does). Conversely, suppose Z meets U, say at $s \in Z \cap U$. But now $s^- \subseteq Z$, so that $s^- \in \mathcal{Z}$, and $s^- \in \natural(U)$, to show that $\mathcal{Z}$ meets $\natural(U)$.

The set $\mathcal{Z}$ is closed irreducible in $\mathsf{sob}(S)$, and hence is non-empty. This gives some $X \in \mathcal{Z}$ with X closed irreducible in S. In particular, $X \subseteq Z$ and X is non-empty, to show that Z is non-empty. (Equivalently, since $\mathcal{Z}$ meets $\natural(S)$ we know Z meets S, and hence Z is non-empty.)

Now suppose Z meets both U and V (from $\mathcal{O}S$). Then $\mathcal{Z}$ meets both $\natural(U)$ and $\natural(V)$ so that (since $\mathcal{Z}$ is closed irreducible)

$$\mathcal{Z} \text{ meets } \natural(U) \cap \natural(V) = \natural(U \cap V)$$

and hence Z meets $U \cap V$. This shows that Z is closed irreducible in S.

Finally, for $X \in \mathsf{sob}(S)$ we have

$$X \in \mathcal{Z} \iff X \subseteq Z \iff X \sqsubseteq Z$$

to show that $\mathcal{Z}$ is the closure of Z in $\mathsf{sob}(S)$.

(d) This is just Lemma 3.2.

(e) If S and $\mathsf{sob}(S)$ are homeomorphic then S is sober by part (c).

Conversely, suppose S is sober. Then, by construction of $\mathsf{sob}(S)$, the assignment is a bijection. For each $s \in S$ and $U \in \mathcal{O}S$, we have

$$s \in U \iff s^{-} \text{ meets } U \iff s^{-} \in \natural(U)$$

to show that the assignment is a homeomorphism. ■

The functorial properties of the construction $\mathsf{sob}(\cdot)$ are best encapsulated by the following factorization result.

3.9 THEOREM. *For each continuous map*

$$S \xrightarrow{\ \phi\ } T$$

from a space S to a sober space T, there is a unique continuous map

$$\mathsf{sob}(S) \xrightarrow{\ \phi^{\sharp}\ } T$$

such that

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\varsigma"'] & & T \\ & \mathsf{sob}(S) \arrow[ur, "\phi^{\sharp}"'] & \end{tikzcd}$$

commutes.

Proof. As usual with a result such as this, we must show there is at most one fill-in arrow, and at least one fill-in arrow. Unusually, both part need a little bit of thought.

For the uniqueness we first make an observation about ς. We show that

$$\varsigma^{\leftarrow}(\natural(U)) = U$$

for $U \in \mathcal{O}S$. (In fact, $\varsigma^{\leftarrow}$ is the inverse of $\natural(\cdot)$.) For each $s \in S$ we have

$$s \in \varsigma^{\leftarrow}(\natural(U)) \iff s^{-} \in \natural(U) \iff s^{-} \text{ meets } U \iff s \in U$$

to give the observation.

Now suppose there is a parallel pair

$$\begin{tikzcd} \mathsf{sob}(S) \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\theta"'] & T \end{tikzcd}$$

of continuous maps such that

$$\psi \circ \varsigma = \phi = \theta \circ \varsigma$$

hold. For each $W \in \mathcal{O}T$ the sets $\psi^{\leftarrow}(W)$ and $\theta^{\leftarrow}(W)$ are open in $\mathsf{sob}(S)$, and hence

$$\psi^{\leftarrow}(W) = \natural(U) \qquad \theta^{\leftarrow}(W) = \natural(V)$$

for some $U, V \in \mathcal{O}S$. But now

$$U = \varsigma^{\leftarrow}(\natural(U)) = (\varsigma^{\leftarrow} \circ \psi^{\leftarrow})(W) = \phi^{\leftarrow}(W) = (\varsigma^{\leftarrow} \circ \theta^{\leftarrow})(W) = \varsigma^{\leftarrow}(\natural(V)) = V$$

and hence

$$\psi^{\leftarrow}(W) = \natural(U) = \natural(V) = \theta^{\leftarrow}(W)$$

holds.

Now, by way of contradiction, suppose $\psi \neq \theta$. We have $\psi(X) \neq \theta(X)$ for some $X \in \mathsf{sob}(S)$. Since T is T_0 there is some $W \in \mathcal{O}T$ which contains exactly one of these points, say

$$\psi(X) \in W \qquad \theta(X) \notin W$$

(by symmetry). But now

$$X \in \psi^{\leftarrow}(W) \qquad X \notin \theta^{\leftarrow}(W)$$

which is the contradiction.

To show the existence of a fill-in arrow it is worth remembering that the continuous map induces an adjoint pair

$$\begin{tikzcd} \mathcal{O}S \arrow[r, shift right, "\phi_*"'] & T \arrow[l, shift right, "\phi^*"'] \end{tikzcd}$$

where

$$\phi^*(V) = \phi^{\leftarrow}(V) \qquad \phi_*(U) = \phi^{\rightarrow}(U')^{-\prime}$$

for $V \in \mathcal{O}T$ and $U \in \mathcal{O}S$.

For $X \in \mathsf{sob}(S)$ consider the closed set

$$\phi^{\rightarrow}(X)^{-}$$

of T. We have

$$\phi^{\rightarrow}(X)^{-} \text{ meets } V \iff \phi^{\rightarrow}(X) \text{ meets } V \iff X \text{ meets } \phi^{\leftarrow}(V)$$

for each $V \in \mathcal{O}T$. Using this a couple of simple arguments shows that $\phi^{\rightarrow}(X)^{-}$ is closed irreducible in T. Since T is sober this set has a unique generic point. Let this be $\phi^{\sharp}(X)$. Thus we have a function

$$\phi^{\sharp} : \mathsf{sob}(S) \longrightarrow T$$

such that

$$\phi^{\rightarrow}(X)^{-} = \phi^{\sharp}(X)^{-}$$

for each $X \in \mathsf{sob}(S)$.

For $X \in \mathsf{sob}(S)$ and $V \in \mathcal{O}T$ we have

$$\phi^{\sharp}(X) \in V \iff \phi^{\sharp}(X)^{-} \text{ meets } V \iff \phi^{\rightarrow}(X)^{-} \text{ meets } V \iff X \text{ meets } \phi^{\leftarrow}(V)$$

so that

$$X \in \phi^{\sharp\leftarrow}(V) \iff X \in \natural(\phi^{\leftarrow}(V))$$

to show

$$\phi^{\sharp\leftarrow}(V) = \natural(\phi^{\leftarrow}(V))$$

and hence $\phi^{\sharp}$ is continuous.

With $X = s^{-}$ for $s \in S$ we have

$$\phi^{\sharp}(s^{-}) \in V \iff s^{-} \text{ meets } \phi^{\leftarrow}(V) \iff s \in \phi^{\leftarrow}(V) \iff \phi(s) \in V$$

for each $V \in \mathcal{O}T$. Since T is $T-0$, this gives

$$(\phi^{\sharp} \circ \varsigma)(s) = \phi^{\sharp}(s^{-}) = \phi(s)$$

to show that the triangle does commute. ■

You probably need a drink after all that.

4 The ideal completion of a poset

More often than not this example is not described in topological terms and, in consequence, can look a bit odd. Here I will first give a brief account of the non-topological version and then show how it is an example of a sober reflection.

4.1 DEFINITION. Let S be a poset.

A subset $X \subseteq S$ is directed if $X \neq \emptyset$ and for each $x, y \in X$ there is some $z \in X$ with $x, y \leq z$.

The poset S is directedly complete if $\bigvee X$ exists (in S) for each directed $X \subseteq S$. ■

We consider how we might convert an arbitrary poset into a directedly complete poset in some 'universal' manner. Of course, the precise meaning of 'universal' is given using standard categorical ideas.

We need the appropriate morphisms

$$S \xrightarrow{\phi} T$$

between, on the one hand, arbitrary posets and, on the other hand, directedly complete posets.

4.2 DEFINITION. A monotone map ϕ between posets S, T (as above) is a function, as indicated, such that

$$x \leq y \Longrightarrow \phi(x) \leq \phi(y)$$

for $x, y \in S$.

Let ***Pos*** be the category of posets and monotone maps.

A d-continuous map ϕ between directedly complete, posets S, T (as above) is a monotone map, as indicated, such that

$$\phi(\bigvee X) = \bigvee \phi^{\rightarrow}(X)$$

for each directed $X \subseteq S$.

Let ***Dpo*** be the category of directedly complete posets and d-continuous maps. ■

Strictly speaking we should check that ***Pos*** and ***Dpo*** are categories, that is both monotone maps and d-continuous maps are closed under composition. However, that is more or less trivial. The only thing to remember is that for a monotone map ϕ, if X is a directed subset of the source, then $\phi^{\rightarrow}(X)$ is a directed subset of the target.

You may wonder why the word 'continuous' is used here. In fact, a function ϕ between directedly complete posets is d-continuous in the sense of Definition 4.2 precisely when it is continuous relative to certain carried topologies. Furthermore, a function ϕ between arbitrary posets is monotone precisely when it is continuous relative to certain other carried topologies. We look at this later.

In the definition of d-continuous we stated explicitly that the map should be monotone. However, a simple argument shows that this is ensured by the continuity property.

[A directedly complete poset is often called a domain. However, the use of that word for this notion is not one of the better ideas that someone has had.]

We have two categories and an obvious forgetful functor

$$\boldsymbol{Pos} \longleftarrow \boldsymbol{Dpo}$$

which forgets the completeness properties of a directedly complete poset and forgets the continuity property of a d-continuous map. We show that this functor has a left adjoint, that is we describe a reflection of ***Pos*** into ***Dpo***.

As you read this next bit you may want to keep an eye on Section 5 of [3].

4.3 DEFINITION. Let S be a poset.

A lower section of S is a subset $L \subseteq S$ such that

$$y \leq x \in L \Longrightarrow y \in L$$

(for $x, y \in S$).

Let $\mathcal{L}S$ be the poset of all lower sections of S under inclusion.

An ideal of S is a lower section $I \in \mathcal{L}S$ which is also directed.

Let $\mathcal{I}S$ be the poset of all ideals of S under inclusion. ■

For each $a \in S$ (a poset) let

$$\eta(a) = \downarrow a = \{x \in S \mid x \leq a\}$$

to obtain the principal ideal generated by a. It is routine to check that the assignment

$$S \xrightarrow{\eta} \mathcal{I}S$$

is monotone. We show that η has even better properties.

The poset $\mathcal{L}S$ is closed under arbitrary unions and intersections, and hence $\mathcal{L}S$ is a complete lattice. This is not the case for $\mathcal{I}S$. In fact, $\mathcal{I}S$ need not be closed under binary intersections, as the following example shows.

4.4 EXAMPLE. Consider the following 4-element poset.

$$\begin{tikzcd} a \arrow[d, dash] \arrow[dr, dash] & b \arrow[d, dash] \arrow[dl, dash] \\ p & q \end{tikzcd}$$

We have

$$\downarrow a = \{a, p, q\} \qquad \downarrow b = \{b, p, q\}$$

whereas

$$\downarrow a \cap \downarrow b = \{p, q\}$$

and this is not an ideal. ■

In spite of this the poset $\mathcal{I}S$ does have sufficient completeness to do a job for us.

4.5 LEMMA. *For each poset S the associated poset $\mathcal{I}S$ is closed under directed unions, and so is a directedly complete poset.*

Proof. Let $\mathcal{X}$ be a directed subfamily of $\mathcal{I}S$ (that is directed in $\mathcal{I}S$). The union $\bigcup \mathcal{X}$ is a lower section of S, so it suffices to show that it is directed in S.

To this end consider $x, y \in \bigcup \mathcal{X}$. We have

$$x \in X \in \mathcal{X} \qquad y \in Y \in \mathcal{X}$$

for some ideal X, Y. Since $\mathcal{X}$ is directed (in $\mathcal{I}S$) there is some $Z \in \mathcal{X}$ with $X, Y \subseteq Z$. But now $x, y \in Z$ and Z is directed (in S) to give some

$$x, y \leq z \in Z \subseteq \bigcup \mathcal{X}$$

as required. ■

In other words, for each poset S the poset $\mathcal{I}S$ of ideals is an object of ***Dpo***. We show it is the reflection of S in ***Dpo***.

4.6 LEMMA. *Let S be an arbitrary poset and let*

$$\begin{tikzcd} \mathcal{I}S \arrow[r, shift left, "\psi"] \arrow[r, shift right, "\theta"'] & T \end{tikzcd}$$

*be a parallel pair of **Dpo**-arrows to a **Dpo**-object T. If*

$$\psi \circ \eta = \theta \circ \eta$$

then $\psi = \theta$.

Proof. Consider any $X \in \mathcal{I}S$ and let

$$\mathcal{X} = \eta^{\rightarrow}(X) = \{\downarrow x \mid x \in X\}$$

to obtain $\mathcal{X} \subseteq \mathcal{I}S$ such that

$$X = \bigcup \mathcal{X}$$

holds. Since X is an ideal, and hence directed in S, we see that $\mathcal{X}$ is directed in $\mathcal{I}S$. Thus

$$\psi(X) = \psi(\bigcup \mathcal{X}) = \bigvee \psi^{\rightarrow}(\mathcal{X}) = \bigvee (\psi \circ \eta)^{\rightarrow}(X)$$

to give

$$\psi(X) = \bigvee (\psi \circ \eta)^{\rightarrow}(X) \qquad \theta(X) = \bigvee (\theta \circ \eta)^{\rightarrow}(X)$$

where the right hand equality follows by a similar argument. Finally, if $\psi \circ \eta = \theta \circ \eta$ then

$$\psi(X) = \bigvee (\psi \circ \eta)^{\rightarrow}(X) = \bigvee (\theta \circ \eta)^{\rightarrow}(X) = \theta(X)$$

and hence $\psi = \theta$, as required. ■

In the terminology of Definition 5.3 of [3], this says that the arrow η is ***Dpo***-epic. This simple observation provides the uniqueness part of the reflection result.

4.7 THEOREM. *For each monotone map*

$$S \xrightarrow{\ \phi\ } T$$

from a poset S to a directedly complete poset T, there is a unique d-continuous map

$$\mathcal{I}S \xrightarrow{\ \phi^{\sharp}\ } T$$

such that

$$\begin{tikzcd} S \arrow[rr, "\phi"] \arrow[dr, "\eta"'] & & T \\ & \mathcal{I}S \arrow[ur, "\phi^{\sharp}"'] & \end{tikzcd}$$

commutes.

Proof. By Lemma 4.6 there is at most one such map $\phi^{\sharp}$.

Given an ideal $X \in \mathcal{I}S$ we find that $\phi^{\rightarrow}(X)$ is directed in T, and hence we may set

$$\phi^{\sharp}(X) = \bigvee \phi^{\rightarrow}(X)$$

to obtain a function $\phi^{\sharp} : \mathcal{I}S \longrightarrow T$. A few calculations show that this is a fill in. ■

This result is the reason why the monotone map

$$S \xrightarrow{\ \eta\ } \mathcal{I}S$$

is called the ideal completion of the poset S.

Out next job is to describe the arrows of ***Pos*** and ***Dpo*** in topological terms.

4.8 DEFINITION. Let S be a poset. Let ΥS be the family of all upper section of S. This is the Alexandroff topology on S.

Let S be a directedly complete poset. Let $\mathcal{O}S$ be the family of all those upper section U of S such that

$$\bigvee X \in U \Longrightarrow X \text{ meets } U$$

holds for each directed subset X of S. This is the Scott topology on S. ■

Strictly speaking before we make this definition we should check that each of ΥS and $\mathcal{O}S$ is a topology on S. The first of these is immediate and the following is the only non-trivial part of the second.

4.9 LEMMA. *Let S be a directedly complete poset. Then $\mathcal{O}S$ is closed under binary intersections.*

Proof. Consider $U, V \in \mathcal{O}S$. Both of these are upper sections of S, hence so is $U \cap V$. Now suppose

$$\bigvee X \in U \cap V$$

for some directed subset $X \subseteq S$. We must show that $X \cap U \cap V$ is non-empty.

We have

$$\bigvee X \in U \qquad \bigvee X \in V$$

so that, since $U, V \in \mathcal{O}S$ we have

$$x \in X \cap U \qquad y \in X \cap V$$

for some $x, y \in S$. Since $x, y \in X$ and X is directed we have some $x, y \leq z \in X$. But U and V are upper sections so $z \in U$ (via x) and $y \in V$ (via y). Thus

$$z \in X \cap U \cap V$$

so that X meets $U \cap V$, as required. ■

It is always useful to have a description of the closed sets of a space. For ΥS these are exactly the lower sections of the poset S. For $\mathcal{O}S$ the closed sets are a bit more interesting. The proof of the following is little more than taking the contrapositive.

4.10 LEMMA. *Let S be a directedly complete poset. Then $\mathcal{C}S$ is the family of the lower sections Z which are closed under directed suprema, that is*

$$X \subseteq Z \Longrightarrow \bigvee X \in Z$$

for each directed subset X.

This description of the Scott topology in terms of the closed sets is often more useful than the official definition in terms of open sets.

With these notions we can give a characterization of the arrows

$$S \xrightarrow{\ \phi\ } T$$

of ***Pos*** and ***Dpo***.

4.11 LEMMA. *A function ϕ between posets S, T (as above) is monotone precisely when it is continuous relative to the two carried Alexandroff topologies.*

A function ϕ between directedly complete posets S, T (as above) is d-continuous precisely when it is continuous relative to the two carried Scott topologies.

Proof. Consider first the arbitrary poset case.

Suppose that ϕ is monotone (from S to T), and consider any $V \in \Upsilon T$. We must show that $\phi^{\leftarrow}(V) \in \Upsilon S$, in other words we must show that $\phi^{\leftarrow}(V)$ is a upper section of S. To this end consider $x \in \phi^{\leftarrow}(V)$ and $x \leq y$ (in S). Then $\phi(x) \in V$ and $\phi(x) \leq \phi(y)$ (since ϕ is monotone), to give $\phi(y) \in V$ (since V is an upper section of S), and hence $y \in \phi^{\leftarrow}(V)$, as required.

Next suppose that ϕ is continuous relative to the two A-topologies, and consider $x \leq y$ in S. The set $V = \uparrow\phi(x)$ is in ΥT, so that $\phi^{\leftarrow}(V) \in \Upsilon S$, that is $\phi^{\leftarrow}(V)$ and upper section of S. But $\phi(x) \in V$, so that $x \in \phi^{\leftarrow}(V)$, to give $y \in \phi^{\leftarrow}(V)$, and hence $\phi(y) \in V$, as required.

Next we consider the directedly complete poset case. Thus we assume that S and T are directedly complete.

Suppose that ϕ is d-continuous (from S to T), and consider any $V \in \mathcal{O}T$. We must show that $\phi^{\leftarrow}(V) \in \mathcal{O}S$. Since each d-continuous map is monotone, this set is certainly an upper section of S. Consider any directed subset X of S with $\bigvee X \in \phi^{\leftarrow}(V)$. Then

$$\bigvee \phi^{\rightarrow}(X) = \phi(\bigvee X) \in V$$

so that $\phi^{\rightarrow}(X)$ meets V (since $\phi^{\rightarrow}(X)$ is directed in T and $V \in \mathcal{O}T$). This gives some $x \in X$ with $\phi(x) \in V$, and hence X meets $\phi^{\leftarrow}(V)$ (at x), to give the required result.

Finally suppose that ϕ is continuous relative to the two Scott topologies. A simple argument (as in the first part) shows that ϕ is monotone. Consider any directed subset X of S. The monotonicity gives

$$\bigvee \phi^{\rightarrow}(X) \leq \phi(\bigvee X)$$

so it suffices to verify the converse comparison. Given a directed subset X of S let

$$c = \bigvee \phi^{\rightarrow}(X)$$

so that $\downarrow c \in \mathcal{C}S$ and hence

$$Z = \phi^{\leftarrow}(\downarrow b) \in \mathcal{C}S$$

(since ϕ is continuous). But $X \subseteq Z$, so that $\bigvee X \in Z$ (by Lemma 4.10), and hence

$$\phi(\bigvee X) \leq c = \bigvee \phi^{\rightarrow}(X)$$

as required. ■

This result shows that both ***Pos*** and ***Dpo*** are subcategories of ***Top***. In particular, when viewed as a space each poset S has a sober reflection $\mathsf{sob}(S)$, as described in Section 3. We already know what this is.

4.12 LEMMA. *Let S be a poset with A-topology ΥS. The closed irreducible subsets of this space are precisely the ideals of S.*

Proof. Suppose first that $X \subseteq S$ is closed irreducible in S. Then X is certainly a lower section of S (since it is a closed subset), and non-empty (since every close irreducible subset is non-empty). Consider $x, y \in X$. Both the open sets $U = \uparrow X$ and $V = \uparrow y$ meet X (at x and y, respectively), and hence X meets $\uparrow x \cap \uparrow y$. This gives some $z \in X$ with $x, y \leq z$, to show that X is directed.

Conversely, suppose X is an ideal of S. Then X is a non-empty lower section of S, and hence is a non-empty closed subset. To show that X is irreducible suppose X meets both of the open sets U and V. This gives some

$$x \in X \cap U \qquad y \in Y \cap V$$

and hence we have some $x, y \leq z \in X$ (since X is directed). But both U and V are upper sections of S, so that $z \in X \cap U \cap V$, to show that X meets $U \cap V$, as required. ■

This result shows that for each poset S we have

$$\mathsf{sob}(S) = \mathcal{I}S$$

as sets. Both of these carry canonical topologies, and we show these are the same topology. On the left we have a the sober reflection topology, that is a typical closed set has the form

$$\natural(L')'$$

for $L \in \mathcal{L}S$ (a typical closed set of S). On the right we have the Scott topology, that is a typical closed set is a lower section of $\mathcal{I}S$ which is closed under unions of directed subfamilies. We have to match the different closed sets.

First of all an unravelling of the definitions gives the following.

4.13 LEMMA. *For each $L \in \mathcal{L}S$ we have*

$$X \in \natural(L')' \iff X \subseteq L$$

for each $X \in \mathcal{I}S$.

This description make the following almost trivial.

4.14 COROLLARY. *For each $L \in \mathcal{L}S$ the family $\natural(L')'$ is closed in $\mathcal{I}S$.*

Proof. Trivially the family $\natural(L')'$ is a lower section of $\mathcal{I}S$. Consider any directed subfamily $\mathcal{X} \subseteq \natural(L')'$. For each $X \in \mathcal{X}$ we have $X \subseteq L$, so that $\bigcup \mathcal{X} \subseteq L$, and hence $\bigcup \mathcal{X} \subseteq \natural(L')'$. Thus $\natural(L')'$ is closed in the S-topology on $\mathcal{I}S$. ■

[You may be wondering where the directedness of $\mathcal{X}$ is used in the last proof. By Lemma 4.5 it ensures that $\bigcup \mathcal{X}$ is an ideal, not just an arbitrary lower section of S.]

The main job now is to show that each S-closed subfamily of $\mathcal{I}S$ has the form $\natural(L')'$ for some $L \in \mathcal{L}S$.

To this end consider $\mathcal{Z} \in \mathcal{C}(\mathcal{I}S)$, that is $\mathcal{Z}$ is a S-closed subfamily of $\mathcal{I}S$. We show that

$$L = \bigcup \mathcal{Z}$$

is the appropriate lower section of S. Since $\mathcal{Z}$ is a family of lower sections of S, we have $L \in \mathcal{L}S$. Also

$$X \in \mathcal{Z} \Longrightarrow X \subseteq L$$

for $X \in \mathcal{I}S$. Thus it suffices to verify the converse of this implication.

4.15 LEMMA. *In the notation above, we have*

$$X \subseteq L \Longrightarrow X \in \mathcal{Z}$$

for each $X \in \mathcal{I}S$.

Proof. Consider any ideal $X \subseteq L$. Let

$$\mathcal{X} = \{{\downarrow}x \mid x \in X\}$$

so that

$$X = \bigcup \mathcal{X}$$

and $\mathcal{X}$ is a family of ideals of S. We show that

$$\mathcal{X} \subseteq \mathcal{Z} \qquad \mathcal{X} \text{ is directed in } \mathcal{I}S$$

so that

$$X = \bigcup \mathcal{X} \in \mathcal{Z}$$

(since $\mathcal{Z}$ is S-closed in $\mathcal{I}S$).

Consider any member ${\downarrow}x$ of $\mathcal{X}$. Thus

$$x \in X \subseteq L = \bigcup \mathcal{Z}$$

to give some ideal $Z \in \mathcal{Z}$ such that $x \in Z$. But now

$${\downarrow}x \subseteq Z \in \mathcal{Z}$$

and $\mathcal{Z}$ is a lower section of $\mathcal{I}S$, so that ${\downarrow}x \in \mathcal{Z}$, to verify the left hand requirement.

To show that $\mathcal{X}$ is directed consider any pair ${\downarrow}x, {\downarrow}y$ of members of $\mathcal{X}$. Then $x, y \in X$ and we remember that X is an ideal, so that we have some $x, y \leq z \in X$, and hence

$${\downarrow}x, {\downarrow}y \subseteq {\downarrow}z \in \mathcal{X}$$

to give the required result. ■

The result of this section are not central to frame theory, but they do give a small illustration of how frames can support lots of different things.

5 The spectrum of a lattice

For the second example we look at the result which is the precursor of much that happens in frame theory, namely the Stone representation of a d-lattice. In Subsection 5.3 of [3] we describe the reflection

$$D \longmapsto \mathcal{I}D$$

form ***Dlt*** to ***Frm***. For each d-lattice D the poset $\mathcal{I}D$ if all ideals of D is a frame, and the assignment

$$\begin{array}{ccc} D & \xrightarrow{\eta_D} & \mathcal{I}D \\ a & \longmapsto & \downarrow a \end{array}$$

is a lattice embedding. Furthermore, η_D has a certain universal property that ensures it is the reflection of D into ***Frm***.

In this section we show how this is related to the Stone representation of D, and to various other frame theoretic constructions.

Let us recall the salient features of Stone's result.

Let D be a d-lattice. We attach to D a certain topological space

$$\mathsf{spec}(D)$$

the spectrum of D, and we set up an embedding

$$D \xrightarrow{\delta} \mathcal{O}\mathsf{spec}(D)$$

into the carried topology to give a concrete representation of D. Here we will first describe the original, more common, construction of $\mathsf{spec}(D)$. Then we will show that

$$\mathsf{spec}(D) = \mathsf{pt}(\mathcal{I}D)$$

(as spaces) and indicate how the embedding δ arises. In a full account we would also characterize the spaces that arise as spectra, and characterize the range of δ and so describe D as a family of sets under union and intersection. However, that would take us too far away from our central topic.

[The spaces that arise in this way are, quite naturally, called spectral spaces. These spaces also arise as the spectra of commutative rings. This coincidence is more than a curiosity, but that is a story for another day.]

I suppose we should start at the beginning. It will help if you have Subsection 5.3 of [3] to hand, and later we will look at Block 5.5.3.

A d-lattice is a bounded distributive lattice. We fix such a d-lattice D throughout this section. Recall that an ideal of D is a non-empty lower section I of D such that

$$x, y \in I \Longrightarrow x \vee y \in I$$

for $x, y \in D$. An ideal I is proper if $\top \notin I$. An ideal I is prime if it is proper and

$$x \wedge y \in I \Longrightarrow x \in I \text{ or } y \in Y$$

(for $x, y \in D$).

With these we can describe one version of the construction of the spectrum

5.1 DEFINITION. let D be a d-lattice.
Let $\mathsf{spec}(D)$ be the set of prime ideals P of D.
For each element $a \in D$ we use

$$P \in U_D(a) \Longleftrightarrow a \notin P$$

(for $P \in \mathsf{spec}(D)$) to extract a subset of $\mathsf{spec}(D)$. ■

The following is the essence of the embedding δ.

5.2 LEMMA. *For each d-lattice D we have*

$$\begin{array}{cc} U_D(\top) = \mathsf{spec}(D) & U(\bot) = \emptyset \\ U_D(a \wedge b) = U_D(a) \cap U_D(b) & U_D(a \vee b) = U_D(a) \cup U_D(b) \end{array}$$

for each $a, b \in D$.

Proof. The top two equalities are immediate.
For the bottom left, from the primeness of $P \in \mathsf{spec}(D)$ we have

$$P \notin U_D(a \wedge b) \Longleftrightarrow a \wedge b \in P \Longleftrightarrow a \in P \text{ or } b \in P \Longleftrightarrow P \notin U_D(a) \text{ or } P \notin U_D(b)$$

which, after taking the contrapositive, gives the equality.
For the bottom right, since $P \in \mathsf{spec}(D)$ is an ideal we have

$$P \notin U_D(a \vee b) \Longleftrightarrow a \vee b \in P \Longleftrightarrow a \in P \text{ and } b \in P \Longleftrightarrow P \notin U_D(a) \text{ and } P \notin U_D(b)$$

which, after taking the contrapositive, gives the equality. ■

This result shows that

$$U_D^{\rightarrow}(D) = \{U_D(a) \mid a \in D\}$$

is a base for a topology on $\mathsf{spec}(D)$. This is the one that we want.

5.3 DEFINITION. For each d-lattice D the set $\mathsf{spec}(D)$ furnished with the topology generated by $U_D^{\rightarrow}(D)$ is the $\mathsf{spectrum}$ of D. ■

Lemma 5.2 also gives the following.

5.4 COROLLARY. *For each d-lattice D the assignment*

$$D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

is a lattice morphism.

This result does not claim that U_D is surjective. That is because, in general, it isn't. The result does not claim that U_D is injective, even though it is. The proof of this requires an axiom of choice in the form of a separation principle. The following is proved in the usual way as an application of Zorn's Lemma.

5.5 Separation Principle. *Let $a \in D$ be an element and let $I \in \mathcal{I}D$ be an ideal of the d-lattice D, and suppose $a \notin I$. Then*

$$a \notin P \qquad I \subseteq P$$

for some prime ideal P of D.

If you think you have seen something like this before, then have a look at item 5.28 of [3]. We will look at the connection between the two later.

A simple application of this principle shows that the morphism U_D is an embedding.

5.6 LEMMA. *For each d-lattice D with associated morphism*

$$D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

we have

$$U_D(a) \subseteq U_D(b) \Longrightarrow a \leq b$$

for each $a, b \in D$. In particular, U_D is an embedding.

Proof. In fact, we prove the contrapositive. Thus consider elements $a, b \in D$ with $a \not\leq b$, Then $a \notin {\downarrow}b$ and so, by the Separation Principle 5.5 we have

$$a \notin P \qquad b \in P$$

for some prime ideal P of D. But now

$$P \in U_D(a) \qquad P \notin U_D(b)$$

to give $U_D(a) \nsubseteq U_D(b)$, as required. ■

We now have two embeddings

$$D \xrightarrow{\eta_D} \mathcal{I}D \qquad\qquad D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

of D into a frame, where the right hand one is clearly spatial. What the connection between these? To answer that we extend the second one and use the same notation.

5.7 DEFINITION. For a d-lattice D let

$$\mathcal{I}D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

be the assignment given by

$$P \in U_D(I) \Longleftrightarrow I \nsubseteq P$$

for each $P \in \mathsf{spec}(D)$ and $I \in \mathcal{I}D$. ■

Actually, we are jumping the gun a little here. Certainly the definition produces a subset $U_D(I) \subseteq \mathsf{spec}(D)$, but how do we now this is open? It's because

$$U_D(I) = \bigcup \{U_D(a) \mid a \in I\}$$

and this right hand side is open. Notice also that for $a \in D$ we have

$$U_D({\downarrow}a) = U_D(a)$$

so that the two uses of 'U_D' will not cause too much confusion.

5.8 LEMMA. *For a d-lattice D the assignment*

$$\mathcal{I}D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

is a surjective frame morphism.

Proof. Trivially, the assignment is monotone, and a simple argument gives

$$U_D(I) \cap U_D(J) = U_D(I \cap J)$$

for $I, J \in \mathcal{I}D$.

Consider any $\mathcal{J} \subseteq \mathcal{I}D$. We verify that

$$U_D(\bigvee \mathcal{J}) \subseteq \bigcup \{U_D(J) \mid J \in \mathcal{J}\}$$

which more or less shows that U_D is a frame morphism. (We shouldn't forget the extremes.)

Consider any

$$P \in U_D(\bigvee \mathcal{J})$$

so we must produce some $J \in \mathcal{J}\}$ with $P \in U_D(J)$. To do that we remember how $\bigvee \mathcal{J}$ is computed in $\mathcal{I}D$.

We have

$$\bigvee \mathcal{J} \nsubseteq P$$

to give some $a \in D$ with

$$a \in \bigvee \mathcal{J} \qquad a \notin P$$

and hence, by the left hand condition, we have

$$a \leq b_1 \vee \cdots \vee b_n \qquad b_i \in J_i \in \mathcal{J}$$

for some selection from $\mathcal{J}$ and members of these. Since $a \notin P$ we have some $b_i \notin P$ to give $J_i \nsubseteq P$, and hence

$$P \in U_D(J_i) \subseteq \bigcup \{U_D(J) \mid J \in \mathcal{J}\}$$

as required.

It remains to show that the assignment is surjective. A typical open set of $\mathsf{spec}(D)$ has the form

$$U_D(X) = \bigcup \{U_D(x) \mid x \in X\}$$

for a subset $X \subseteq D$. An easy calculation shows that

$$U_D(X) = U_D(\langle X \rangle)$$

where $\langle X \rangle$ is the ideal of D generated by X. ■

We have a surjective frame morphism, so the obvious next step is to locate the kernel. We can do better than that. The proof of the following extends that of Lemma 5.6.

5.9 THEOREM. *For a d-lattice D the assignment*

$$\mathcal{I}D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

is a frame isomorphism.

Proof. Given Lemma 5.8 it suffices to show

$$U_D(J) \subseteq U_D(I) \Longrightarrow J \subseteq I$$

for ideals I, J of D. To do that we modify the argument of the proof of Lemma 5.6. As there, we prove the contrapositive.

Suppose $J \nsubseteq I$. There is some element a with

$$a \in J \qquad a \notin I$$

and hence a use of the Separation Principle 5.5 gives

$$a \notin P \qquad I \subseteq P$$

for some prime ideal P. But now

$$J \nsubseteq P \qquad I \subseteq P$$

so that

$$P \in U_D(J) \qquad P \notin U_D(I)$$

for the required result. ■

You may have notice a certain similarity between the constructions of

$$\mathsf{spec}(D) \qquad \mathsf{pt}(A)$$

the spectrum of a d-lattice and the point space of a frame. In fact, the two constructions can be done in parallel, and even viewed as two instances of a single construction. Everything is generated using the schizophrenic object $\mathsf{2}$. We can illustrate that using Theorem 5.9.

For each d-lattice D the two frames

$$\mathcal{I}D \qquad \mathcal{O}\mathsf{spec}(D)$$

are isomorphic, and so have essentially the same point space. That of $\mathcal{O}\mathsf{spec}(D)$ is just the parent space $\mathsf{spec}(D)$ (since this is sober). This must also be the point space of $\mathcal{I}D$. We can verify this directly.

Remember that the points of $\mathcal{I}D$ are those ideals which are $\cap$-irreducible in $\mathcal{I}D$.

5.10 LEMMA. *For each d-lattice D we have*

$$\mathsf{spec}(D) = \mathsf{pt}(\mathcal{I}D)$$

(as sets).

Proof. Consider $P \in \mathsf{spec}(D$, so we must show that P is $\cap$-irreducible in $\mathcal{I}D$. Certainly $P \neq D$, since P is proper. Thus we must verify

$$I \cap J \subseteq P \Longrightarrow I \subseteq P \text{ or } J \subseteq P$$

for ideals $I, J \in \mathcal{I}D$. To do this we use the contrapositive. Thus suppose

$$I \nsubseteq P \text{ and } J \nsubseteq P$$

so that

$$x \in I, x \notin P \qquad y \in J, y \notin P$$

for some $x, y \in D$. Since P is prime this gives $z = x \wedge y \notin P$. But $z \leq x, y$, so that $z \in I$ and $z \in J$, and hence $I \cap J \nsubseteq P$ (as witnessed by z).

Conversely, consider $P \in \mathsf{pt}(\mathcal{I}D)$, so we must show that P is prime. Certainly $P \neq D$, so we must verify

$$x \wedge y \in P \Longrightarrow x \in P \text{ or } y \in P$$

for $x, y \in D$. For such x, y let

$$I = \downarrow x \qquad J = \downarrow y$$

so that

$$I \cap J = \downarrow (x \wedge y)$$

and hence

$$x \wedge y \in P \Longrightarrow I \cap J \subseteq P \Longrightarrow I \subseteq P \text{ or } J \subseteq P \Longrightarrow x \in P \text{ or } y \in P$$

as required. ■

The d-lattice D has associated morphisms

$$D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D) \qquad \mathcal{I}D \xrightarrow{U_{\mathcal{I}D}} \mathcal{O}\mathsf{pt}(\mathcal{I}D)$$

(in the appropriate category) to a pair of topologies on the same set $\mathsf{spec}(D) = \mathsf{pt}(\mathcal{I}D)$. These are given by

$$P \in U_D(a) \Longleftrightarrow a \notin P \qquad P \in U_{\mathcal{I}D}(I) \Longleftrightarrow I \nsubseteq P$$

for $a \in D, I \in \mathcal{I}D$, and P a prime ideal of D. The morphism $U_{\mathcal{I}D}$ is surjective, but U_D need not be.

From earlier we know that the left hand morphism can be factorized as

$$D \xrightarrow{\eta} \mathcal{I}D \xrightarrow{U_D} \mathcal{O}\mathsf{spec}(D)$$

where this U_D is an isomorphism. By comparing the definition of this U_D with that of $U_{\mathcal{I}D}$ we see that the two assignments are exactly the same. Thus we have the following.

5.11 THEOREM. *For each d-lattice D the spectrum $\mathsf{spec}(D)$ is nothing more than the point space $\mathsf{pt}(\mathcal{I}D)$ of the frame of ideals of D. Furthermore, the composite*

$$D \xrightarrow{\eta} \mathcal{I}D \xrightarrow{U_{\mathcal{I}D}} \mathcal{O}\mathsf{pt}(\mathcal{I}D)$$

is the representation of D.

To conclude this section we can give a bit more information to explain the reflection of block 5.3.3 of [3]. The crucial result there is the existence of a unique frame morphism

$$\Phi A \xrightarrow{\iota_A} \mathcal{I}A$$

for each frame A. In fact, such a morphism exists for any d-lattice, not just a frame. (The frame properties are needed to produce a frame morphism

$$\mathcal{I}A \xrightarrow{\zeta_A} A$$

which is composed with ι_A.)

For any d-lattice D we take

$$T = \boldsymbol{Set}[D, 2] \qquad S = \boldsymbol{Dlt}[D, 2]$$

the set of characteristic functions on D and the set of frame characters on D. Using the sierpinski topology on 2 we furnish T with the product topology and then take the subspace topology on S. In particular, restriction

$$\begin{array}{ccc} \mathcal{O}T & \longrightarrow & \mathcal{O}S \\ U & \longmapsto & U \cap S \end{array}$$

is a surjective frame morphism.

Next we observe there is a bijective correspondence

$$\begin{array}{ccc} \boldsymbol{Dlt}[D, 2] & \cong & \mathsf{spec}(D) \\ p & \longleftrightarrow & P \end{array}$$

given by

$$p(x) = 0 \iff x \in P$$

for $x \in A$. We check that this is a homeomorphism relative to the two carried topologies (the restricted product topology and the spectral topology).

All this produces a morphism

$$\Phi D = \boldsymbol{Set}[D, 2] \longrightarrow \mathcal{O}\boldsymbol{Dlt}[D, 2] \cong \mathcal{O}\mathsf{spec}(D) \cong \mathcal{I}D$$

passing through two isomorphisms. Tracking through these we find that this is the morphism ι extended to the lattice case. Essentially, this is what is being proved towards the end of Block 5.5.3 of [3].

6 Frames with no points

In Section 1 we indicated that there are frame with no points. In particular we observed that an atomless complete boolean algebra is a frame with no points (since the points of a boolean frame are just its maximal elements, the complements of its atoms). The main aim of this section is to show there is at least one quite exotic frame with no points. However, before we do that let's fill in a gap. Let's see where we can find atomless complete boolean algebras.

To do that we remember where we can find complete boolean algebras.

Consider any space S with is topology $\mathcal{O}S$. By Example 4.10 of [3] we know that the quotient frame $(\mathcal{O}S)_{\neg\neg}$ is a complete boolean algebra and is just the algebra of regular open sets of S. We will show that for a suitable space S this algebra has no points.

We use a simple observation.

Consider an arbitrary quotient

$$A \xrightarrow{j^*} A_j$$

of a frame A as determined by a nucleus j on A. What are the points of this quotient? By viewing a point as a frame character we see that each point of the quotient

$$A \xrightarrow{j^*} A_j \longrightarrow 2$$

produces a point of the parent, by composition. What is this in terms of $\wedge$-irreducible elements? I will leave the proof of the following as an exercise.

6.1 LEMMA. *For each frame quotient*

$$A \xrightarrow{j^*} A_j$$

as $\wedge$-irreducible elements, the points of A_j is are precisely those points p of A that are fixed by j, that is $j(p) = p$.

With this we can generate lots of atomless complete boolean algebras.

6.2 EXAMPLE. Let S be a space and assume that S is both T_1 and sober. In particular, any T_2 space will do. Consider the canonical quotient

$$\mathcal{O}S \longrightarrow (\mathcal{O}S)_{\neg\neg}$$

to the complete boolean algebra of regular open sets of S.

Consider any point of $(\mathcal{O}S)_{\neg\neg}$. By Lemma 6.1 this must also be a point of $\mathcal{O}S$. Since S is sober it must have the form $s^{-\prime}$ for some $s \in S$. Since S is T_1, this must have the form $\{s\}'$ for some $s \in S$.

This is certainly a point of $\mathcal{O}S$. To be a point of $(\mathcal{O}S)_{\neg\neg}$ it must satisfy

$$\neg\neg(\{s\}') = \{s\}'$$

that is

$$\{s\}'^{-\circ} = \{s\}'$$

which gives

$$\{s\}^{\circ -} = \{s\}$$

by taking the complement of both sides.

Suppose we have such a point s, and let $U = \{s\}^\circ$. Then $U \neq \emptyset$, for otherwise $\{s\} = U^- = \emptyset$, which is not so. Consider any $t \in U$. Then $t \in U^- = \{s\}$, to show that $t = s$. Thus we have $U = \{s\}$, and hence

$$S \cap U = \{s\}$$

to show that s is an isolated point of the whole space.

This shows that any T_1+sober space with no isolated points provides an example of an atomless complete boolean algebra, and hence a frame with no points. ■

This is all very nice but what if every frame without points is boolean.

As I said, the main purpose of this section is to exhibit an exotic frame with no points. We construct a frame Ω with the following properties.

- Ω has no points, that is $\mathsf{pt}(\Omega) = \emptyset$.
- Ω has many regular elements, that is elements a with $\neg\neg a = a$.
- Ω has many dense elements, that is elements a with $\neg a = \bot$.

As we have seen, it is easy to produce a frame with the first two properties, for we simply take a complete, atomless boolean algebra. The third property is not so straight forward. The frame we produce is also complicated in other ways, but these are not easy to describe in a succinct fashion.

To construct this exotic frame Ω we start from a certain poset S and take an appropriate quotient of the frame ΥS of upper sections of S. In other words, Ω is the family of certain upper sections of S; it is a fixed set of ΥS.

Recall that for such a quotient Ω the implication on Ω agrees with that on ΥS. Furthermore, we have

$$s \in (V \supset U) \Longleftrightarrow \uparrow s \cap V \subseteq U$$

for each $U, V \in \Upsilon S$ and $s \in S$. We arrange that $\emptyset \in \Omega$, so that negation

$$\neg(\cdot) = (\cdot) \supset \emptyset$$

on Ω agrees with that on ΥS. This helps us get at the regular and dense elements of Ω.

Each point, $\cap$-irreducible element, of Ω arises from a point of ΥS. We arrange that the quotient

$$\Upsilon S \longrightarrow \Omega$$

kills all the points of ΥS, that is no point survives the passage to Ω.

We use a particular poset S which has certain extra structure and extra properties which are important for the construction. This means there are two possible ways to proceed.

We could take a concrete approach. We say at the outset what the particular poset S is, and then work with this throughout. In this approach the distinction between relevant and the irrelevant properties of S may not be clear.

We could take an abstract approach. We postulate a poset with certain properties, use these to produce Ω, and then give an examples of such a poset. With this approach the distinction between the general and the particular is clearer.

Here I will take the abstract approach. However, once I have set up the basic data I will tell you what the particular example is, so you may keep this in mind as the abstract development unfolds. Of course, later we will look again in more detail at the particular example.

Let S be a poset. We let r, s, t range over S. We write

$$r|s$$

to indicate that r, s are incompatible in S, that is there is no t with $r, s \leq t$.

(C1) $s \not\leq t \Longrightarrow (\exists r)[t \leq r \text{ and } r|s]$

(C2) $s \in S, \sigma \in \Sigma - S \Longrightarrow (\exists r)[s \leq r \text{ and } r|\sigma]$

(C3) $(\forall \sigma \in \Sigma)(\exists s \in S)[s \leq \sigma]$

(C4) $r, s \leq \sigma \Longrightarrow (\exists t)[r, s \leq t \leq \sigma]$

(C5) If $R \subseteq S$ is directed in S then $\bigvee R$ exists in Σ.

(C6) If the lower section $R \subseteq S$ is directed in S and if $s \leq \bigvee R$, then $s \in R$.

Table 1: Conditions on the pair $S \subseteq \Sigma$

We assume S sits inside a larger poset Σ. We let ρ, σ, τ range over Σ. This fount convention will help us to distinguish between S and Σ.

We assume the comparison on S is just the restriction of the comparison on Σ. Thus we write $\leq$ for both comparisons. It turns out that

$$(\cdot|\cdot) \quad r|s \text{ (in } S) \Longrightarrow r|s \text{ (in } \Sigma)$$

also holds. In fact, in the particular example, for elements $r, s \leq \sigma$ the join $r \vee s$ exists in S, but this partial join operation on S is not important in the construction.

We need certain restriction on S and Σ. These are listed in Table 1, but before we look at those let me tell you what the particular example is.

Let A and B be infinite sets with $|A| < |B|$. Let Σ be the set of all *partial* functions from A to B, ordered by extension. Let S be the set of all *finite* partial functions from A to B. We will return to this example after the general development.

On an historical point, I have no idea where the example originally came from. I found it in some of my very old handwritten notes. The poset S is clearly a family of forcing conditions, as used in Set Theory. However, I know very little about that subject, so it's a bit of a mystery how I got hold of the example. Perhaps some of you out there recognize it from elsewhere.

Consider a pair of posets, $S \subseteq \Sigma$, as above. We assume the conditions (C1 – C6) of Table 1. Observe that (C4) give the condition $(\cdot|\cdot)$ above. We use these conditions to verify certain properties of ΥS.

6.3 LEMMA. *Condition (C1) ensures that for each $s \in S$ the principal upper section $\uparrow s$ of S is regular in ΥS.*

Proof. We require

$$\neg\neg(\uparrow s) \subseteq \uparrow s$$

for the given $s \in S$. Observe that

$$r \in \neg(\uparrow s) \Longleftrightarrow \uparrow r \cap \uparrow s = \emptyset \Longleftrightarrow r|s$$

for each $r \in S$. Consider any $t \in \neg\neg(\uparrow s)$. Thus

$$\uparrow t \cap \neg(\uparrow s) = \emptyset$$

and we require $s \leq t$.

If $s \nleq t$ then (C1) gives

$$t \leq r \qquad r|s$$

for some $r \in S$. But then

$$r \in {\uparrow}t \cap \neg({\uparrow}s)$$

which is impossible. ■

When we construct the quotient Ω of ΥS (with $\emptyset \in \Omega$) we ensure that ${\uparrow}s \in \Omega$ for each $s \in S$. This will gives many regular elements of Ω.

We also want many dense elements. Consider $\sigma \in \Sigma$. We use

$$r \in D(\sigma) \Longleftrightarrow r|\sigma$$

(for $r \in S$) to extract an upper section of S.

6.4 LEMMA. *Condition (C2) ensures that for each $\sigma \in \Sigma - S$ the upper section $D(\sigma)$ of S is dense in ΥS.*

Proof. Let $D = D(\sigma)$. We require $\neg D = \emptyset$.

Consider any $s \in S$. Condition (C2) provides some $r \in S$ with

$$s \leq r \qquad r|\sigma$$

that is with

$$r \in {\uparrow}s \cap D$$

so that

$${\uparrow}s \cap D = \emptyset$$

and hence $s \notin \neg D$, as required. ■

When we construct the quotient Ω of ΥS (with $\emptyset \in \Omega$) we ensure that $D(\sigma) \in \Omega$ for many (but not all) $\sigma \in \Sigma - S$. This gives us many dense elements of Ω.

We want to kill the points of ΥS. To do that we need to know what these points look like. Consider $\sigma \in \Sigma$. We use

$$r \in P(\sigma) \Longleftrightarrow r \nleq \sigma$$

(for $r \in S$) to extract an upper section of S.

6.5 LEMMA. *Conditions (C3) and (C4) ensure that for each $\sigma \in \Sigma$ the upper section $P(\sigma)$ of S is $\cap$-irreducible in ΥS.*

Proof. Let $P = P(\sigma)$. By (C3) there is some $s \notin P$, so that $P \neq S$.

Consider $U, V \in \Upsilon S$ with

$$U \nsubseteq P \qquad V \nsubseteq P$$

so we require $U \cap V \nsubseteq P$. We have some

$$r \in U \quad r \leq \sigma \qquad s \in V \quad s \leq \sigma$$

and then (C4) gives some

$$r, s \le t \le \sigma$$

so that

$$t \in U \cap V \quad t \notin P$$

as required. ■

This result gives us many points of ΥS. We need to know that every point arises in this way.

6.6 LEMMA. *Suppose $P \in \Upsilon S$ is $\cap$-irreducible. Conditions (C5) and (C6) ensure that $P = P(\sigma)$ for some $\sigma \in \Sigma$.*

Proof. For the given P let $R = S - P$ to obtain a lower section of S. For $r, s \in S$ we have

$$r, s \in R \Longrightarrow \uparrow r, \uparrow s \not\subseteq P \Longrightarrow \uparrow r \cap \uparrow s \not\subseteq P \Longrightarrow (\exists t \in R)[r, s \le t]$$

to show that R is directed. By condition (C5) we have an element

$$\sigma = \bigvee R$$

of Σ. We show that $P = P(\sigma)$.

Since

$$r \notin P \Longrightarrow r \in R \Longrightarrow r \le \sigma \Longrightarrow r \notin P(\sigma)$$

we have $P(\sigma) \subseteq P$.

Conversely, if $s \notin P(\sigma)$, that is

$$s \le \sigma = \bigvee R$$

then condition (C6) gives $s \in R$, that is $s \notin P$. Thus $P \subseteq P(\sigma)$. ■

Our next job is to produce a suitable quotient of of ΥS. To do that we first describe a general method of producing quotients of ΥS for an arbitrary poset S. For this method the conditions of Table 1 play no role and we do not need a larger poset Σ. Once we have this general method we look at the particular example indicated earlier.

Thus, for the time being let S be an arbitrary poset.

Recall that an interval of S is a subset $H \subseteq S$ with

$$\left.\begin{array}{c} r \le s \le t \\ r, t \in H \end{array}\right\} \Longrightarrow s \in H$$

for all $r, s, t \in S$. Sometimes an interval is called a convex part. Note that each upper section and each lower section of S is an interval. Note also that the intersection of a family of intervals is itself an interval.

To obtain a quotient of ΥS we use a family of intervals with a suitable property. Since such a family does not occur elsewhere in these notes we need not give it a memorable name.

6.7 DEFINITION. A family $\mathcal{H}$ of intervals of a poset S is suitable if

$$\uparrow s \in \mathcal{H} \qquad \left.\begin{array}{r} H \in \mathcal{H} \\ H \subseteq K \end{array}\right\} \Longrightarrow K \in \mathcal{H}$$

for each $s \in S$ and all interval H, K. ■

Trivially, the family of all intervals of S is suitable. Eventually we use a much smaller suitable family and apply the following construction.

6.8 DEFINITION. Let $\mathcal{H}$ be a suitable family of intervals of the poset S. The induced family $\Omega(\mathcal{H}) \subseteq \Upsilon S$ is given by

$$U \in \Omega(\mathcal{H}) \Longleftrightarrow (\forall s \in S)[s \notin U \Longrightarrow \uparrow s - U \in \mathcal{H}]$$

(for $U \in \Upsilon S$). ■

The following result is, perhaps, a surprise.

6.9 LEMMA. *Let $\mathcal{H}$ be a suitable family of intervals of the poset S. The induced family $\Omega(\mathcal{H})$ is a fixed set of ΥS, and $\emptyset \in \Omega(\mathcal{H})$.*

Proof. Let $\Omega = \Omega(\mathcal{H})$. We must show the following.

(i) $\emptyset \in \Omega$

(ii) Ω is closed under arbitrary intersections.

(iii) if $U \in \Omega$ and $V \in \Upsilon S$, then $(V \supset U) \in \Omega$.

We deal with each of these in turn.

(i) For each $s \in S$ we have

$$\uparrow s - \emptyset = \uparrow s \in \mathcal{H}$$

by the first part of suitability.

(ii) Consider any $\mathcal{U} \subseteq \Omega$, and let $V = \bigcap \mathcal{U}$. To show $V \in \Omega$ consider any $s \notin V$, and let

$$K = \uparrow s - V$$

so that $K \in \mathcal{H}$ is required. We produce some $H \in \mathcal{H}$ with $H \subseteq K$, and then invoke the second part of suitability.

Since $s \notin V$ we have some $U \in \mathcal{U} \subseteq \Omega$ with $s \notin U$. Since $U \in \Omega$ we have

$$H = \uparrow s - U \in \mathcal{H}$$

and hence an inclusion $H \subseteq K$ will suffice. Since $V \subseteq U$ we have $U' \subseteq V'$, and hence $H \subseteq K$.

(iii) Consider any

$$U \in \Omega \qquad V \in \Upsilon S$$

and let

$$W = (V \supset U)$$

so that we require $W \in \Omega$. Remember that

$$r \in W \iff \uparrow r \cap V \subseteq U$$

(for $r \in S$).

To show $W \in \Omega$ consider any $s \notin W$, and let

$$K = \uparrow s - W$$

so that $K \in \mathcal{H}$ is required. We produce some $H \in \mathcal{H}$ with $H \subseteq K$, and then invoke the second part of suitability.

Since $s \notin W$ we have

$$\uparrow s \cap V \nsubseteq U$$

to produce some $s \leq t \in V$ with $t \notin U$. Since $U \in \Omega$ we have

$$H = \uparrow t - U \in \mathcal{H}$$

and hence an inclusion $H \subseteq K$ will suffice.

Consider any $r \in H$. We have

$$s \leq t \leq r \qquad r \notin U$$

so that $r \in V$ (since $t \in V$) and

$$\uparrow r \cap V \nsubseteq U$$

(as witnessed by r). This non-inclusion gives $r \notin W$, and hence $r \in K$. ■

Since the family $\Omega(\mathcal{H})$ is a fixed set of ΥS, it is determined by some nucleus on ΥS. I do not know a simple description of this nucleus.

This conclude the abstract generalities. We can now begin the particular concrete construction.

We need a pair of poset $S \subseteq \Sigma$ satisfying the conditions of Table 1. Recall what these ensure.

(C1) Each $\uparrow s$ is regular in ΥS (for $s \in S$).

(C2) Each $D(\sigma)$ is dense in ΥS (for $\sigma \in \Sigma - S$).

(C3,4) Each $P(\sigma)$ is a point of ΥS (for $\sigma \in \Sigma$).

(C5,6) Each point of ΥS has the form $P(\sigma)$ (for some $\sigma \in \Sigma$).

We use the construction of Definition 6.8 to produce a quotient Ω of ΥS. We ensure the following.

(D1) Each $\uparrow s \in \Omega$ (for $s \in S$).

(D2) Each $D(\sigma) \in \Omega$ (for many $\sigma \in \Sigma - S$).

(D3) The quotient kills each $P(\sigma)$ (for $\sigma \in \Sigma$).

We choose the posets $S \subseteq \Sigma$ and the family $\mathcal{H}$ with these in mind.

Let A and B be a pair of infinite sets with $|A| < |B|$, that is the cardinality of B is strictly greater than the cardinality of A. A *partial* function from A to B is a function

$$\sigma : X \longrightarrow B$$

for some subset $X \subseteq A$. We write $\partial\sigma$ for the domain of definition of σ. Thus

$$\sigma : \partial\sigma \longrightarrow B$$

and we have a value $\sigma(a)$ for each $a \in \partial\sigma$. These partial functions are partially ordered by extension, thus

$$\sigma \leq \tau \Longleftrightarrow \partial\sigma \subseteq \partial\tau \text{ and } \sigma = \tau|_{\partial\sigma}$$

for such functions σ, τ.

Let Σ be the poset of all these partial functions.

Observe also that

$$\sigma|\tau \Longleftrightarrow (\exists a \in \partial\sigma \cap \partial\tau)[\sigma(a) \neq \tau(a)]$$

for $\sigma, \tau \in \Sigma$. In fact, if σ and τ are compatible, (not $\sigma|\tau$), then there is a smallest common extension $\sigma \cup \tau$ of the two.

Notice that since $|A| < |B|$, no $\sigma \in \Sigma$ has range B. It is this fact that will enable us to kill all the points of ΥS.

Let S be the subposet of Σ of all *finite* partial functions from A to B. Since A is infinite we see that S is strictly smaller than Σ.

As above we let ρ, σ, τ range over Σ, and let r, s, t range over S.

We must check the conditions (C1 – C6) for this pair.

(C1) Consider $s, t \in S$ with $s \not\leq t$. There is some $a \in \partial s$ with either $a \notin \partial t$ or $a \in \partial t$ and $s(a) \neq t(a)$. For the second alternative we have $t|s$, so we may take $r = t$. For the first alternative we extend t to

$$r = t \cup [a \mapsto b]$$

for any $b \in B$ with $b \neq s(a)$.

(C2) Consider $s \in S$ and $\sigma \in \Sigma - S$. The set

$$\partial\sigma - \partial s$$

is non-empty (since $\partial\sigma$ is infinite but ∂s is finite). Consider any $a \in \partial\sigma - \partial s$ and extend s to

$$r = s \cup [a \mapsto b]$$

where $b \neq \sigma(a)$.

(C3) Consider any $\sigma \in \Sigma$. For any finite $X \subseteq \partial\sigma$ let $s = \sigma|_X$. Then $s \in S$ and $s \leq \sigma$. (Actually, the empty partial function is in S and is the bottom of σ.)

(C4) Consider $r, s \in S$ and $\sigma \in \Sigma$ with $r, s \leq \sigma$. The two functions r, s are compatible, so we may take $t = r \cup s$.

(C5) Suppose $R \subseteq S$ is directed in S. For each $r_1, \dots, r_m \in R$ there is some $s \in R$ with $r_1, \dots, r_m \leq s$. Consider the subset

$$X = \bigcup \{\partial r \mid r \in R\}$$

of A. For each $a \in X$ there may be several $r \in R$ with $a \in \partial r$. However, the value $r(a)$ is independent of the choice of r. Thus we may define a function

$$\sigma : X \longrightarrow B$$

by

$$\sigma(a) = r(a) \quad \text{for any } r \in R$$

to obtain the smallest common extension of the $r \in R$. This is the required $\bigvee R$.

(C6) Suppose R is a directed lower section of S, and let $\sigma = \bigvee R$. Consider any $s \in S$ with $s \leq \sigma$. The set ∂s is finite. For each $a \in \partial s$ there is some $r \in R$ with $a \in \partial r$, and $s(a) = r(a)$. This gives finitely many $r_1, \dots, r_m \in R$ with

$$\partial s \subseteq \partial r_1 \cup \cdots \cup \partial r_m$$

and for each $a \in \partial s$ there is some r_i with $s(a) = r_i(a)$. But R is directed, so there is some $r \in R$ with $r_1, \dots, r_m \leq r$. This gives some $s \leq r$, and hence $s \in R$.

To conclude the construction we need a suitable family of intervals of S. This, of course, must ensure that (D1, D2, D3) hold.

6.10 DEFINITION. An interval H of S is collectively surjective if for each $b \in B$ there is some $s \in H$ and $a \in \partial s$ with $s(a) = b$.

Let $\mathcal{H}$ be the family of collectively surjective interval. ■

Each $\sigma \in \Sigma$ is not surjective. Thus the set

$$S - P(\sigma) = \{s \in S \mid s \leq \sigma\}$$

is not collectively surjective. This indicates that a collectively surjective interval has to be comparatively large.

6.11 LEMMA. *The family $\mathcal{H}$ of collectively surjective intervals is suitable.*

Proof. Consider any $s \in S$ and $b \in B$. The set A is infinite and ∂s is finite, so there is some $a \in A - \partial s$. The extension

$$t = s \cup [a \mapsto b]$$

ensures that $\uparrow s \in \mathcal{H}$.

The second requirement is trivial. ■

This result with Lemma 6.9 gives us a quotient $\Omega = \Omega(\mathcal{H})$ of ΥS with $\emptyset \in \Omega$. It remains to verify (D1, D2, D3) for this frame Ω.

6.12 LEMMA. *For each $s \in S$ we have $\uparrow s \in \Omega$.*

Proof. Consider $t \notin \uparrow s$, that is with $s \not\leq t$. We require $\uparrow t - \uparrow s \in \mathcal{H}$. Consider any $b \in B$. We produce an extension

$$r = t \cup [a \mapsto b]$$

of t with $s \not\leq r$. Of course, we have to take a little care with out choice of $a \in A$.

Since $s \not\leq t$ there is some $c \in A$ where one of the two conditions

$$c \in \partial s \quad c \notin \partial t \qquad\qquad c \in \partial s \cap \partial t \quad s(c) \neq t(c)$$

holds. We consider any element $a \in A$ with

$$a \notin \partial t \cup \{c\}$$

which we can do since $\partial t \cup \{c\}$ is finite and A is infinite.

Since $a \notin \partial t$, the function r exists. Thus it suffices to show $s \not\leq r$.

By way of contradiction suppose $s \leq r$. There are two cases corresponding to the left and the right conditions on c given above.

For the left hand case we have

$$c \in \partial s \subseteq \partial r = \partial t \cup \{a\}$$

so that $c = a$. But we have chosen $a \neq c$.

For the right hand case we have

$$s(c) = r(c) = t(c)$$

which contradicts the choice of c. ■

This result with Lemma 6.3 gives us many regular elements of Ω. In a similar way we use Lemma 6.4 to produce many dense elements of Ω.

6.13 LEMMA. *Consider any $\sigma \in \Sigma - S$ where $A - \partial\sigma$ is infinite. Then $D(\sigma) \in \Omega$.*

Proof. Consider any $s \notin D(\sigma)$, that is with σ and s compatible, so that

$$\tau = \sigma \cup s$$

exists in Σ. We require $\uparrow s - D(\sigma) \in \mathcal{H}$.

Consider any $b \in B$. We produce an extension

$$\sigma \leq \tau \leq \tau \cup [a \mapsto b]$$

and then take

$$t = s \cup [a \mapsto b] \leq \tau \cup [a \mapsto b]$$

to get $t \in \uparrow s - D(\sigma)$.

We have

$$\partial\tau = \partial\sigma \cup \partial s$$

with $A - \partial\sigma$ infinite and ∂s finite. Thus

$$A - \partial\tau \neq \emptyset$$

and we may take any $a \in A - \partial\tau$. ■

There are many $\sigma \in \Sigma - S$ with $A - \partial\sigma$ infinite. Thus, by Lemma 6.4, there are many dense elements of Ω.

At last we have got to the point of the whole construction.

6.14 THEOREM. *The frame Ω has no points.*

Proof. Consider any point P of ΥS. We want $P \notin \Omega$. For this is suffices to exhibit

$$s \notin P \qquad \uparrow s - P \notin \mathcal{H}$$

for some $s \in S$.

By Lemma 6.6 we have $P = P(\sigma)$ for some $\sigma \in \Sigma$. Consider any $s \in S$ with $s \leq \sigma$. We have $s \notin P$.

By way of contradiction suppose $\uparrow s - P \in \mathcal{H}$. We have

$$t \in \uparrow s - P \Longleftrightarrow s \leq t \leq \sigma$$

(for $t \in S$), so that σ is surjective (since the family of all such t is collectively surjective). But $|\partial\sigma| \leq |A| < |B|$, and hence σ is not surjective. ■

You might think that a frame without points can be constructed starting with any frame. Let us say a nucleus j on a frame A kills all the points if $j(p) = \top$ for each point p. For such a nucleus j the quotient A_j has no points. For instance, at the beginning of this section we saw that for a T_1+sober space S with no isolated points the double negation nucleus kills all the points.

If the nucleus j kills all the points of its parent frame A, then A_j has no points, but it may not be a very interesting frame. To get round that we can try to make j as small as possible. Thus let k be the infimum of all the nuclei which kill all the points, and look at A_k. Unfortunately it can happen that A_k has some points. There is something to investigate here and a story to be told. But not just now.

7 Frames with enough points

Each frame A has a point space $S = \mathsf{pt}(A)$ together with a canonical surjective morphism

$$A \xrightarrow{U_A} \mathcal{O}S$$

which indexes the topology on S. By Theorem 3.1 this morphism is the spatial reflection of A, that is $\mathsf{pt}(A)$ is the universal way of converting A into a space.

We have seen, in Section 6, that some frame have no points. There are some quite large and complicated frames with no points. In other words, in general, the morphism U_A can be a long way from being injective.

7.1 DEFINITION. A frame A is spatial if its associated morphism U_A is injective. ■

As usual, we view the points of a frame A as its $\wedge$-irreducible elements. This gives us a useful characterization of spatiality.

7.2 LEMMA. *A frame A with $S = \mathsf{pt}(A)$ is spatial precisely when*

$$a \not\leq b \Longrightarrow (\exists s \in S)[a \not\leq s \text{ and } b \leq s]$$

for each $a, b \in A$.

Proof. The frame A is spatial precisely when U_A is injective, that is

$$U_A(a) \subseteq U_A(b) \Longrightarrow a \leq b$$

for $a, b \in A$. The left hand side of this implication unravels as

$$(\forall s \in S)[a \not\leq s \Longrightarrow b \not\leq s]$$

so the required implication is the contrapositive of the given one. ■

Some frames are 'obviously' spatial. For instance, let S be any space and consider its topology $\mathcal{O}S$. This frame is 'obviously' spatial. The slight catch is the the points of $\mathcal{O}S$ are not the points of S. We have a continuous map

$$\begin{array}{ccc} S & \longrightarrow & \mathsf{pt}(\mathcal{O}S) \\ s & \longmapsto & s^{-\prime} \end{array}$$

but this may be neither injective nor surjective.

Observe that for $s \in S$ and $U \in \mathcal{O}S$ we have

$$s \notin U \Longleftrightarrow s \in U' \Longleftrightarrow s^{-} \subseteq U' \Longleftrightarrow U \subseteq s^{-\prime}$$

and this gives the following.

7.3 LEMMA. *For each space S the topology $\mathcal{O}S$ is spatial.*

Proof. We use Lemma 7.2. Consider any $U, V \in \mathcal{O}S$ with $U \not\subseteq V$. There is some $s \in U - V$. But now, by the observation above, we have

$$U \not\subseteq s^{-\prime} \qquad V \subseteq s^{-\prime}$$

and $s^{-\prime}$ is a point of $\mathcal{O}S$. ■

A space S provides some, but perhaps not all, of the points of $\mathcal{O}S$. However, it does provide enough points to ensure that $\mathcal{O}S$ is spatial, and S doesn't need to be sober to do this. There's a thought.

Informally, we sometimes say that a frame A has enough points if it is spatial. Sometimes a frame does have enough points but, at first sight, it is not all clear where the points come from. Let's look at a simple example of this.

7.4 LEMMA. *Each finite frame is spatial.*

Proof. Let A be a finite frame and consider a pair of elements a, b with $a \not\leq b$. We must separate these by a $\wedge$-irreducible element.

Consider $Z \subseteq A$ given by

$$z \in Z \Longleftrightarrow a \not\leq z \text{ and } b \leq z$$

(for $z \in A$). We know $Z \neq \emptyset$, since $b \in Z$. It suffices to show that Z contains a $\wedge$-irreducible element.

The set Z is finite (since A is finite). Thus Z has a maximal member. By that we mean a member that is maximal in Z, not necessarily maximal in A. Let m be such a maximal member. We show that m is $\wedge$-irreducible.

Since $a \not\leq m$ we have $m \neq \top$.

Consider $x, y \in A$ with $x \not\leq m$ and $y \not\leq m$. We require $x \wedge y \not\leq m$.

Since

$$x \not\leq m \qquad y \not\leq m$$

we have

$$m < m \vee x \qquad m < m \vee y$$

and hence

$$m \vee x \notin Z \qquad m \vee y \notin Z$$

by the maximality of m. Since

$$b \leq m \leq m \vee x \qquad b \leq m \leq m \vee y$$

this gives

$$a \leq m \vee x \qquad a \leq m \vee y$$

and hence

$$a \leq (m \vee x) \wedge (m \vee y) = m \vee (x \wedge y)$$

(since A is distributive). Thus, if $x \wedge y \leq m$ then $a \leq m$, which is not so. ■

You may think I have made heavy weather of this proof, and perhaps I have, but I have done it for a purpose. The proof is a template for many proof of spatiality. First we maximize something or other, and then we show that this maximal thingy has a prime-like property. This gives us the separating point that we want.

The crucial problem is that often the maximizing step is not just a use of finite cardinalities. It often needs a choice principal, and often this is a variant of Zorn's Lemma.

7.5 ZORN'S LEMMA. *Let S be a poset in which each directed subset has an upper bound. Then each element of S lies below a maximal element.*

Usually ZL is stated using upper bounds of chains. However, more often than not this 'directed' version is easier to use. Note that the upper bound need not be a supremum, although it often is in practice.

For the next example of spatiality we return to the Stone representation theorem of Section 5.

Let D be a d-lattice, let $\mathcal{I}D$ be its frame of ideals, and let $\mathsf{spec}(D)$ be its space of prime ideals. By Lemma 5.10 we know that $\mathsf{spec}(D)$ is precisely the set $\mathsf{pt}(\mathcal{I}D)$ of points of $\mathcal{I}D$. Furthermore, we observed that the two carried topologies are precisely the same. Thus, as in Theorem 5.11, we have a commuting triangle

$$\begin{tikzcd} & \mathcal{I}D \arrow[dd, "U_D"] \\ D \arrow[ur, "\eta"] \arrow[dr, "\delta"'] & \\ & \mathcal{O}\mathsf{spec}(D) \end{tikzcd}$$

where δ is the traditional, point-sensitive, representation of D, and η is the point-free version. The right hand vertical arrow, U_D, is just the spatial reflection of $\mathcal{I}D$.

By Theorem 5.9 we know that U_D is an isomorphism. In other words, this result shows that $\mathcal{I}D$ is spatial. That proof makes use of the Separation Property 5.5. It is instructive to look at a slightly different proof which makes use of ZL.

To help with a couple of applications we need the following notion.

7.6 DEFINITION. An element d of a frame A is compact if

$$d \le \bigvee X \Longrightarrow d \in X$$

for each directed lower section X of A. ■

For instance, if you think about it, an open set U of a space is compact in $\mathcal{O}S$ precisely when it is compact in the usual topological sense.

7.7 LEMMA. *Let A be a frame, let $d \in A$, and let $\mathcal{Z}(d)$ be the poset of all ideals Z of A with $d \notin Z$.*

(a) The poset $\mathcal{Z}(d)$ is closed under unions of directed subfamilies.

(b) Each maximal member of $\mathcal{Z}(d)$ is a prime ideal.

(c) If d is compact then each maximal member of $\mathcal{Z}(d)$ is principal.

Proof. (a) It suffices to observe that the union of a directed family of ideals is itself an ideal.

(b) Consider a maximal member M of $\mathcal{Z}(d)$. Since $d \notin M$, we see that M is proper. To show that M is prime consider $x, y \in A$ with $x \notin M$ and $y \notin M$. We require $x \wedge y \notin M$.

Since

$$x \notin M \qquad y \notin M$$

we have

$$M \subsetneq M \vee {\downarrow}x \qquad M \subsetneq M \vee {\downarrow}y$$

using the ideals generated by $M \cup \{x\}$ and $M \cup \{y\}$, respectively. By the maximality of M we have

$$d \in M \vee {\downarrow}x \qquad d \in M \vee {\downarrow}y$$

and hence

$$d \le m_1 \vee x \qquad d \le m_2 \vee y$$

for some $m_1, m_2 \in M$. Let

$$m = m_1 \vee m_2 \in M$$

so that

$$d \leq m \vee x \qquad d \leq m \vee y$$

and hence

$$d \leq (m \vee x) \wedge (m \vee y) = m \vee (x \wedge y)$$

(since D is distributive). Thus, if $x \wedge y \in M$ then $d \in M$, which is not so.

(c) Let M be a maximal member of $\mathcal{Z}(d)$. This ideal MV is directed in A, and so $p = \bigvee M$ exists in A. We have $M \subseteq {\downarrow}p$, so It sufficed to show $p \in M$.

By way of contradiction, suppose $p \in M$. Thus $M \subsetneq {\downarrow}p$ and hence $d \in {\downarrow}p$ by the maximality of M. Thus

$$d \leq \bigvee M$$

and hence $d \in M$ (since d is compact and M is a directed lower section). This is the contradiction. ■

Part (a) of this result shows that we may apply ZL to $\mathcal{Z}(d)$. Thus each member of $\mathcal{Z}(d)$ lies below a maximal member which, by part (b), is a prime ideal of A. Part (c) shows that under certain circumstances that prime ideal has the form ${\downarrow}p$ for some $p \in A$. That element p is $\wedge$-irreducible, and so is a point of A.

As a first example let's see how this idea generalizes that of the proof of Theorem 5.9.

7.8 EXAMPLE. In this example you need to keep control of yourself, for we look at ideals on two different levels.

Let D be a d-lattice, and let $A = \mathcal{I}D$ be its frame of ideals. We also need to look at the poset of ideals of A.

Consider any $a \in D$ and any $I \in A$ with $a \notin I$. We produce a prime ideal P of D with $a \notin P$ and $I \subseteq P$. Of course, this prime ideal of D is $\cap$-irreducible in A, so we are looking for a point of A.

Observe that ${\downarrow}a$ is a compact member of A. We look at $\mathcal{Z}({\downarrow}a)$, which is a family of ideals of (not in) A.

Let

$$\Downarrow I = \{J \in A \mid J \subseteq I\}$$

the set of ideals J of D with $J \subseteq I$. This is a principal ideal of A. Also

$${\downarrow}a \in \Downarrow I \iff {\downarrow}a \subseteq I \iff a \in I$$

so that $\Downarrow I \in \mathcal{Z}({\downarrow}a)$.

By ZL there is some maximal member M of $\mathcal{Z}({\downarrow}a)$ with $\Downarrow I \subseteq M$. By Lemma 7.7 this M is a principal prime ideal of A, and so it has the form $\Downarrow P$ for some $\cap$-irreducible member P of A. This is a prime ideal of D. We have both

$${\downarrow}a \notin \Downarrow P \qquad \Downarrow I \subseteq \Downarrow P$$

which unravel to

$$a \notin P \qquad I \subseteq P$$

for the required result. ■

This example is a bit hairy because of the two levels of ideals involved, and in truth it is not the neatest proof of the spatiality of $\mathcal{I}D$. But it does illustrate how frame theoretic ideas can set certain results in a wider context.

The next application of Lemma 7.7 deals entirely with frames.

7.9 DEFINITION. A frame A is conjunctive if for all $a, b \in A$ with $a \not\leq b$ we have

$$a \vee z = \top \qquad b \vee z \neq \top$$

for some $z \in A$. ■

This notion is the dual of the disjunctive property of distributive lattices, hence the name. It is a weakening of the fitness property for frames, and so it often called subfitness. Both of these properties are concerned with the block structure of the assembly NA. Some information about this can be found in [2]. In topological terms it is a rather weak separation property, weaker than T_1.

The following result was fist proved in [1].

7.10 THEOREM. *Let A be a conjunctive frame where the top $\top$ is compact. Then A is spatial.*

Proof. Consider element $a \not\leq b$ of A. We must produce a point p of A with $a \neq p$ and $b \leq p$.

Since A is conjunctive we have

$$a \vee z = \top \qquad b \vee z \neq \top$$

for some $z \in A$. Let $d = \top$ and, as in Lemma 7.7, let $\mathcal{Z}(\top)$ be the family of ideals Z with $\top \notin Z$ (that is the family of proper ideals). We have $\downarrow(b \vee z) \in \mathcal{Z}(\top)$.

By Lemma 7.7(a) and ZL there is a maximal member M of $\mathcal{Z}(\top)$ with $b \vee z \in M$. Since $\top$ is compact, Lemma 7.7 ensures that $M = \downarrow p$ for some point p of A. We have

$$b \leq b \vee z \leq p$$

so that $a \neq p$, for otherwise $\top = a \vee z \leq p$, which is not so. ■

For the second application of Lemma 7.7 we again look at a restricted class of frames.

A frame is compactly generated if for all $a, b \in A$ with $a \not\leq b$, there is some compact $d \in A$ with $d \leq a$ and $d \not\leq b$. In other words, each element of A is the supremum of the compact elements below it.

7.11 THEOREM. *Each compactly generated frame A is spatial.*

Proof. Consider elements $a \not\leq b$ of A. We must produce a point p of A with $a \not\leq p$ and $b \leq p$.

Since a is compactly generated there is a compact element $d \in A$ with $d \leq a$ and $d \not\leq b$. As in Lemma 7.7, let $\mathcal{Z}(d)$ be the family of ideals Z with $d \notin Z$. We have $\downarrow(b \in \mathcal{Z}(\top)$.

By Lemma 7.7(a) and ZL there is a maximal member M of $\mathcal{Z}(d)$ with $b \vee z \in M$. Since d is compact, Lemma 7.7 ensures that $M = \downarrow p$ for some point p of A. Thus $b \leq p$. If $a \leq p$, then

$$d \leq a \leq p \in M$$

which is not so. ■

So far in this section we have approached the points of a frame via its prime ideals. But a point also corresponds to a completely prime filter, and this gives us a second way of producing points.

In the following we repeat the definition of completely prime to emphasize the comparison.

7.12 DEFINITION. A filter F on a frame A is, respectively,

open	completely prime

if

$$\bigvee X \in F \Longrightarrow X \text{ meets } F$$

for all

all directed	all

subsets $X \subseteq A$. ■

Often an open filter is said to be 'Scott-open'. This is because each such filter F on A is open in the Scott topology on A. With a little more effort a better terminology could have been devised, but unfortunately the above terminology has stuck.

The following is the filter analogue of Lemma 7.7.

7.13 LEMMA. *For an element d of a frame A let $\mathcal{F}(d)$ be the poset of all open filter F of A with $d \notin F$.*

(a) The poset $\mathcal{F}(d)$ is closed under unions of directed subfamilies.

(b) Each maximal member of $\mathcal{F}(d)$ is a completely prime filter.

Proof. (a) Let $\mathcal{G}$ be a directed subfamily of $\mathcal{F}(d)$. The union $\bigcup \mathcal{G}$ is an ideal, so it suffices to show that F is open. To this end let $X \subseteq A$ be directed in A with $\bigvee X \in F$. We have $\bigvee X \in G$ for some $G \in \mathcal{G}$. But G is open, so X meets G, and hence X meets $\bigcup \mathcal{F}(d)$.

(b) Let M be a maximal member of $\mathcal{F}(d)$. We know that M is open, that is

$$\bigvee X \in M \Longrightarrow X \text{ meets } M$$

for each directed $X \subseteq A$. We require this implication for *all* $X \subseteq A$.

Let X be an arbitrary subset of A. Let Y be the directed closure of X, the set of all elements

$$x_1 \vee \cdots \vee x_k$$

for $x_1, \ldots, x_k \in X$. We have

$$\bigvee Y = \bigvee X \in M$$

and hence Y meets M (since M is open). Thus we have

$$x_1 \vee \cdots \vee x_k \in M$$

for some for $x_1, \ldots, x_k \in X$. We show that $x_i \in M$ for some $1 \leq i \leq k$.

By way of contradiction, suppose $x_i \notin M$ for each $1 \le i \le m$. Thus

$$M \subsetneq M \vee \uparrow x_i$$

and hence the maximality of M gives

$$d \ge m_i \vee x_i$$

for some $m_i \in M$. Let

$$m = m_1 \wedge \cdots \wedge m_k \in M$$

so that

$$d \ge (m_1 \vee x_1) \wedge \cdots \wedge (m_k \vee x_k) = m \wedge (x_1 \vee \cdots \vee x_k) \in M$$

which is the contradiction. ■

We use this result in conjunction with ZL. Of course, the result doesn't tell us that $\mathcal{F}(d)$ is non-empty, so in general there is still some work to be done.

7.14 DEFINITION. A frame A has enough open filters if for each $a, b \in A$ with $a \not\le b$ there is some open filter F with $a \in F$ and $b \notin F$.

There is a stronger property of having enough completely prime filters. This, of course, is equivalent to being spatial, which shows us why the following result is useful.

7.15 THEOREM. *A frame A is spatial precisely when it has enough open filters.*

Proof. Suppose A is spatial. Consider $a, b \in A$ with $a \not\le b$. There is some point p of a with $a \not\le p$ and $b \le p$. Let P be the corresponding completely prime filter, that is the filter given by

$$z \in P \Longleftrightarrow x \not\le p$$

(for $z \in A$). We have $a \in P$ and $b \notin P$, and P is open.

Conversely, suppose A has enough pen filter. Consider $a, b \in A$ with $a \not\le b$. Let $\mathcal{F}(b)$ be the poset of open filter F with $b \notin F$. Since A has enough open filter here is at least one $F \in \mathcal{F}(b)$ with $a \in F$. By Lemma 7.13(a) and ZL there is a maximal member P of $\mathcal{F}(b)$ with $a \in P$. By Lemma 7.13(b) this filter P is completely prime. This leads to the required point separation of a and b. ■

There are several other examples of the spatiality of a frame, but these are best done in the appropriate context.

References

[1] C.H. Dowker and Dona Papert Strauss: Separation axioms for frames, pages 223-240 of Topics in Topology, ed Á. Császár, North Holland (1972).

[2] H. Simmons: Regularity, Fitness, and the Block Structure3 of Frames, Applied categorical Strauctures 14 (2006) 1-34.

The whole collection of notes can be found on my web pages with holding the relevant documents. Here are the first few parts.

[3] The basics of frame theory.

[4] The assembly of a frame.

[5] The point space of a frame.

[6] The fundamental triangle of a space.

[7] Boolean reflections of frames.

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