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The assembly of a frame

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In [11] we met the notions of a frame and a frame morphism. These form the objects and arrows of the category $\boldsymbol{Frm}$. We saw that the universal algebra of a frame A is best done using certain operators on A, the nuclei on A. The collection NA of all these, the assembly of A, is partially ordered by the pointwise comparison. In fact, NA is a complete lattice since it is closed under pointwise infima.

In this document we begin to look at the structure of this lattice NA, and some of the properties of the construction $N(\cdot)$. The two main facts that we learn are as follows.

- The assembly NA of a frame A is itself a frame and the two are connected

$$A \xrightarrow{n_A} NA$$

  by an epic embedding.

- The construction $N(\cdot)$ is an endo-functor on $\boldsymbol{Frm}$, and the embedding n_A is natural for variation of A.

And, of course, there are other little snippets as well.

Contents

1 Inflators on a frame

To analyse the family of all nuclei on a frame it is useful to work inside a larger family of operators, the inflators on the frame. Such inflators are easier to handle and are simpler than nuclei, and they can be used to generate nuclei in various ways. They also pop up in several other places (and similar ideas occur in other parts of mathematics). As well as nuclei we also use three other families of special kinds of inflators.

1.1 DEFINITION. Let A be a frame.

(a) An inflator on A is a function

$$f : A \longrightarrow A$$

which is inflationary and monotone, that is

$$x \leq f(x) \qquad x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x, y \in A$. Let IA be the family of all inflators on A.

(b) An inflator f is stable if

$$f(x) \wedge y \leq f(x \wedge y)$$

for all $x, y \in A$. Let SA be the family of all stable inflators on A.

(c) A pre-nucleus is an inflator f such that

$$f(x) \wedge f(y) \leq f(x \wedge y)$$

for all $x, y \in A$. Let PA be the family of all pre-nuclei on A.

(d) A closure operation is an idempotent inflator, that is an inflator f such that $f^2 = f$ (where $f^2 = f \circ f$). Let CA be the family of all closure operations on A.

(e) A nucleus on A is an idempotent pre-nucleus. Let NA be the family of all nuclei on A.

We call NA the assembly of A. ■

Observe that

$$\text{(NPSI)} \qquad NA \subseteq PA \subseteq SA \subseteq IA$$

and it is not hard to see that these four families are distinct. We will see some particular examples later. Notice also that

$$NA \subseteq CA \subseteq IA$$

with

$$NA = PA \cap CA$$

by definition. The following is the only other useful observation.

1.2 LEMMA. *For each frame A we have*

$$NA = SA \cap CA$$

in other words, each stable closure operation is a nucleus.

Proof. It suffices to show $SA \cap CA \subseteq NA$. To this end consider any stable inflator f. Two uses of stabilitygives

$$f(x) \wedge f(y) \leq f(x \wedge f(y)) \leq f^2(x \wedge y)$$

(for $x, y \in A$). Thus if f is also idempotent, then f is a nucleus. ■

At this point I must say a few words about the terminology.

In the literature the term ‘pre-nucleus’ is used for two different notions; a pre-nucleus asd here, and a stable inflator. Both notions are important, so both need names. I believe th eterminolgy used here is the more common one.

Since each inflator is monotone, each pre-nucleus f satisfies he stronger property

$$f(x) \wedge f(y) = f(x \wedge y)$$

for $x, y \in A$. However, stable inflators do not have a corresponding stronger property.

The five families IA, SA, PA, CA, NA have quite a lot of structure.

1.3 DEFINITION. The inflators on a frame A are compared pointwise, that is

$$f \leq g \Longleftrightarrow (\forall x \in A)[f(x) \leq g(x)]$$

for $f, g \in IA$. ■

This gives us five posets IA, SA, PA, CA, NA. Our aim is to analyse the structure of NA, but along he way we will obtain a bit of information about the other posets. For instance, each of the five posets is complete. To see this we use a simple constuction.

1.4 DEFINITION. For a frame A let $\mathcal{F} \subseteq IA$ be a family of inflators. The pointwise infimum of $\mathcal{F}$ is the function

$$\bigwedge \mathcal{F} : A \longrightarrow A$$

given by

$$\bigwedge \mathcal{F}(x) = \bigwedge \{f(x) \mid f \in \mathcal{F}\}$$

for each $x \in A$. ■

As the following shows, this construction gives us lots of completeness. It also shows that there is no illogicality in the terminology.

1.5 LEMMA. *Let A be a frame. Each of the posets IA, SA, PA, CA, NA is closed under pointwise infima. Each poset is complete, and for each the infima are computed pointwise.*

Proof. We make a series of almost trivial observations.

Consider $\mathcal{F} \subseteq IA$. A few moment’s thought gives $\bigwedge \mathcal{F} \in IA$ (even when $\mathcal{F}$ is empty). Note also that $\bigwedge \mathcal{F} \leq f$ for each $f \in \mathcal{F}$. Also, $g \in IA$ satisfies $g \leq f$ for each $f \in \mathcal{F}$, then $g \leq \bigwedge \mathcal{F}$ by the construction of $\bigwedge \mathcal{F}$. Thus $\bigwedge \mathcal{F}$ is the infimum of $\mathcal{F}$ in IA.

Suppose each member of $\mathcal{F}$ is stable. Then, for each $x, y \in A$ we have

$$(\bigwedge \mathcal{F})(x) \wedge y = \bigwedge \{f(x) \wedge y \mid f \in \mathcal{F}\} \leq \bigwedge \{f(x \wedge y) \mid f \in \mathcal{F}\} = (\bigwedge \mathcal{F})(x \wedge y)$$

to show that $\bigwedge \mathcal{F}$ is stable.

Consider $\mathcal{F} \subseteq PA$. Then, for each $x, y \in A$ we have

$$\begin{aligned} (\bigwedge \mathcal{F})(x) \wedge (\bigwedge \mathcal{F})(y) &= \bigwedge \{g(x) \wedge h(y) \mid g, h \in \mathcal{F}\} \\ &\leq \bigwedge \{f(x) \wedge f(y) \mid f \in \mathcal{F}\} \\ &\leq \bigwedge \{f(x \wedge y) \mid f \in \mathcal{F}\} \qquad = (\bigwedge \mathcal{F})(x \wedge y) \end{aligned}$$

to show that $\bigwedge \mathcal{F} \in PA$.

Suppose each member of $\mathcal{F}$ is idempotent. For each $f \in \mathcal{F}$ and $x \in A$ we have

$$(\bigwedge\mathcal{F})(x) \leq f(x)$$

and $\bigwedge\mathcal{F}$ is monotone, so that

$$(\bigwedge\mathcal{F})^2(x) \leq (\bigwedge\mathcal{F})(f(x)) \leq f^2(x) = f(x)$$

which is enough to show that $\bigwedge\mathcal{F}$ is idempotent.

Various combinations of these observations give all the required results. ■

Since each of the five posets IA, SA, PA, CA, NA has all infima, each is complete, and so each has al suprema. In particular, NA is a complete lattice. However the supremum of a family $\mathcal{J} \subseteq NA$ can be difficult to locate. This is one reason why PA and SA are useful. However, be warned. A supremum of a family computed in one poset may not be the same as the supremum of the same family computed in a different poset. This explains the following notation.

1.6 DEFINITION. For a frame A let $\mathcal{F} \subseteq IA$ be a family of inflators. The pointwise supremum of $\mathcal{F}$ is the function

$$\dot{\bigvee}\mathcal{F} : A \longrightarrow A$$

given by

$$\dot{\bigvee}\mathcal{F}(x) = \bigvee\{f(x) \mid f \in \mathcal{F}\}$$

for each $x \in A$. ■

We must be a little careful with this notion. When $\mathcal{F}$ is empty we have

$$(\dot{\bigvee}\mathcal{F})(x) = \bigvee\emptyset = \bot$$

so that $\dot{\bigvee}\mathcal{F}$ is not an inflator (unless A is trivial). When $\mathcal{F}$ is non-empty the construction works better.

1.7 LEMMA. *Let A be a frame and consider any non-empty family $\mathcal{F} \subseteq IA$ of inflators. Then the pointwise supremum $\dot{\bigvee}\mathcal{F}$ is an inflator, and is the supremum of $\mathcal{F}$ in IA.*

Proof. We have at least one $f \in \mathcal{F}$, and

$$x \leq f(x) \leq (\dot{\bigvee}\mathcal{F})(x)$$

for each such f and $x \in A$. This shows that $\dot{\bigvee}\mathcal{F}$ is inflationary. The monotonicity is immediate. Thus $\dot{\bigvee}\mathcal{F} \in IA$.

We have just seen that $f \leq \dot{\bigvee}\mathcal{F}$ for each $f \in \mathcal{F}$. Conversely, suppose $f \leq g \in IA$ for each $f \in \mathcal{F}$. Then

$$\dot{\bigvee}\mathcal{F}(x) = \bigvee\{f(x) \mid f \in \mathcal{F}\} \leq g(x)$$

for each $x \in A$. Thus $\dot{\bigvee}\mathcal{F}$ is the supremum of $\mathcal{F}$ in IA. ■

There is little content in this result. A similar construction works for any complete lattice. But we can still make good use of it.

Suppose we have $\mathcal{F} \subseteq NA$ (with $\mathcal{F}$ non-empty). We have $NA \subseteq IA$ and, in fact, NA is a sub-poset of IA. Thus $\dot{\bigvee}\mathcal{F} \in IA$, but $\dot{\bigvee}\mathcal{F}$ need not be a nucleus. We know that $\mathcal{F}$ must have a supremum in NA but, as we will see later, this can be much larger than $\dot{\bigvee}\mathcal{F}$. It is the difference between $\dot{\bigvee}\mathcal{F}$ and the supremum in NA that is the cause of most of the difficulties in the analysis of NA. Later on in these notes, in Section 4, we will see how certain suprema can be calculated in NA.

As I said, Lemma 1.7 has little content. Here is a refinement that requires FDL (the Frame Distributive Law).

1.8 LEMMA. *Let A be a frame and consider any non-empty family $\mathcal{F} \subseteq SA$ of inflators. Then the pointwise supremum $\dot{\bigvee}\mathcal{F}$ is stable, and is the supremum of $\mathcal{F}$ in SA.*

Proof. We know that $\dot{\bigvee}\mathcal{F}$ is an inflator, and lies above each member of $\mathcal{F}$.

For each $x, y \in A$ we have

$$\begin{aligned}(\dot{\bigvee}\mathcal{F})(x) \wedge y &= \bigvee\{f(x) \mid f \in \mathcal{F}\} \wedge y \\ &= \bigvee\{f(x) \wedge y \mid f \in \mathcal{F}\} \\ &\leq \bigvee\{f(x \wedge y) \mid f \in \mathcal{F}\} = (\dot{\bigvee}\mathcal{F})(x \wedge y)\end{aligned}$$

to show that $\dot{\bigvee}\mathcal{F}$ is stable. Here the second equality follows by FDL.

The remainder of the proof is immediate. ■

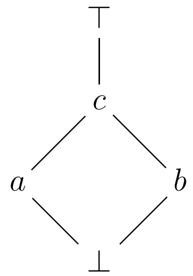
This is looking promising, isn't it? The next thing to check is that PA is closed under pointwise suprema. Unfortunately, this isn't true.

For nuclei j, k on some frame A let $j \mathbin{\dot\vee} k$ be the pointwise join given by

$$(j \mathbin{\dot\vee} k)(x) = j(x) \vee k(x)$$

(for $x \in A$). By Lemma 1.8 we know that $j \mathbin{\dot\vee} k$ is stable, but the following example shows that it need not be a pre-nucleus.

1.9 EXAMPLE. For the 5-element frame



consider the nuclei $j = \mathsf{v}_a, j = \mathsf{v}_b$. Thus

$$j(x) = (a \supset x) \qquad k(x) = (b \supset x)$$

(for each element x). We have

$$j(a) = \top \quad j(\bot) = b \qquad k(b) = \top \quad k(\bot) = a$$

to give

$$(j \dot{\vee} k)(a) = \top \qquad (j \dot{\vee} k)(b) = \top$$

whereas

$$(j \dot{\vee} k)(a \wedge b) = (j \dot{\vee} k)(\bot) = j(\bot) \vee k(\bot) = c$$

and hence $j \dot{\vee} k$ is not a pre-nucleus. ■

In general, suprema in PA are not computed pointwise, but things are not too bad.

Recall that a family $\mathcal{F}$ of inflators is directed if for each $f, g \in \mathcal{F}$ there is some $h \in \mathcal{F}$ with $f, g \leq h$.

1.10 LEMMA. *For each frame A the poset PA is closed under directed pointwise suprema.*

Proof. Let $\mathcal{F}$ be a directed family of pre-nuclei. For each $x, y \in A$ we have

$$\left(\dot{\bigvee}\mathcal{F}\right)(x) \wedge \left(\dot{\bigvee}\mathcal{F}\right)(y) = \bigvee \{f(x) \wedge g(y) \mid f, g \in \mathcal{F}\}$$

by two uses of the FDL. The right hand family certainly contains all the values

$$h(x) \wedge h(y) = h(x \wedge y)$$

for $h \in \mathcal{F}$ (for consider $f = g = h$). But $\mathcal{F}$ is directed, so for each $f, g \in \mathcal{F}$ there is some $h \in \mathcal{F}$ with

$$f(x) \wedge g(y) \leq h(x) \wedge h(y) = h(x \wedge y)$$

and hence

$$\left(\dot{\bigvee}\mathcal{F}\right)(x) \wedge \left(\dot{\bigvee}\mathcal{F}\right)(y) = \bigvee \{h(x \wedge y) \mid h \in \mathcal{F}\} = \left(\dot{\bigvee}\mathcal{F}\right)(x \wedge y)$$

as required. ■

Much of our analysis will use the lattice structure of IA, SA, and PA, but there is also another useful structure lurking around.

The proof of the following is immediate from the definitions involved.

1.11 LEMMA. *For each frame A the three posets IA, SA, PA are closed under composition. That is, of f and g belong to one of the posets then so does $g \circ f$.*

We will use composition of inflators (often of a special kind) quite a lot. Of course, the assembly NA is not closed under composition, but we can still form the composite $k \circ j$ of two nuclei j and k to obtain a pre-nucleus. Working in the larger poset gives us a few extra tricks up our sleeve.

There is something more general going on here. This is certainly worth developing, but will not be done here. We will stick fairly closely to the analysis of frames. However, as a hint of this more general topic observe that composition interacts nicely with the comparison. We have

$$\left.\begin{matrix} f \leq g \\ h \leq k \end{matrix}\right\} \Longrightarrow f \circ h \leq g \circ k$$

and

$$f, g \leq g \circ f$$

for inflators f, g, h, k. The second of these gives a useful way of producing directed families of inflators.

1.12 LEMMA. *For a frame A, if $\mathcal{F}$ is a non-empty family of inflators that is closed under composition then $\mathcal{F}$ is directed.*

For the moment this result is a mere curiousity, but it will become useful in Section 4.

The main aim of this section is to show that the assembly NA of a frame is itself a frame. We know that NA is a complete lattice so, by Lemma 1.5, it suffices to show that NA carries an implication. To do that we use a little trick with stable inflators. The following result appears in a slightly different contexts as Lemma 4.2 of [6] and Lemma 3.1 of [5].

1.13 LEMMA. *Let A be a frame, let $f \in SA$, and let $k \in NA$. There is some $l \in SA$ such that*

$$f \wedge g \leq k \Longleftrightarrow g \leq l$$

for all $g \in SA$. Furthermore, we have $l \in NA$.

Proof. For the given $f \in SA$ and $k \in NA$ let $\mathcal{G}$ be the family of all $g \in SA$ such that $f \wedge g \leq k$. We first show that $\mathcal{G}$ is closed under composition.

To verify this consider $g, h \in \mathcal{G}$. For each $x \in A$ we have

$$\begin{array}{rcll} (f \wedge (g \circ h))(x) & = & f(x) \wedge g(h(x)) & \\ & \leq & f(x) \wedge g(f(x) \wedge h(x)) & \\ & \leq & f(x) \wedge g(k(x)) & \\ & \leq & f(k(x)) \wedge g(k(x)) & \\ & \leq & k^2(x) & = \quad k(x) \end{array}$$

to show $g \circ h \in \mathcal{G}$, as required. Here the second step holds since $g \in SA$, the third holds since $h \in \mathcal{G}$, and the fifth since $g \in \mathcal{G}$.

Let

$$l = \dot{\bigvee} \mathcal{G}$$

to obtain the supremum of $\mathcal{G}$ in SA. For each $x \in A$ we have

$$(f \wedge l)(x) = f(x) \wedge l(x) = f(x) \wedge \bigvee\{g(x) \mid g \in \mathcal{G}\} = \bigvee\{f(x) \wedge g(x) \mid g \in \mathcal{G}\} \leq k(x)$$

to show that $l \in \mathcal{G}$. But now, by the first observation, we have $l^2 = l \circ l \in \mathcal{G}$, to give $l^2 \leq l$, and hence $l \in NA$.

For $g \in SA$ we have

$$f \wedge g \leq k \Longrightarrow g \leq l$$

by construction. Conversely, if $g \leq l$ then

$$f \wedge g \leq f \wedge l \leq k$$

since $l \in \mathcal{G}$, to complete the proof. ∎

With this we can achieve out first objective.

1.14 THEOREM. *For each frame A the assembly NA is also a frame.*

Proof. By Lemma 1.5 the poset NA is complete. Thus by Lemma 1.7 of [11] it suffices to show that NA carries an implication operation.

Consider $j, k \in NA$. By Lemma 1.13 there is some $l \in NA$ such that

$$j \wedge g \leq k \Longleftrightarrow g \leq l$$

for all $g \in PA$. In particular, this is true for $g \in NA$, to show that l is the required implication $(j \supset k)$. ■

This important result was first proved by Dowker and Papert as Theorem 1 of [1]. Unfortunately they worked in terms of congruences (being unaware that frame congruences could be encoded by nuclei). This meant that they partially ordered their analogue of the assembly upside down. If only they had realized this, it would perhaps have prevented a lot of nonsense being published.

2 The frame of all nuclei

Theorem 1.14 is one of the most important result in the whole of frame theory. It gives the subject it own special flavour. For that reason it will do no harm if we look at two other proofs of the same result.

2.1 THEOREM. *For each frame A the poset SA (of all stable inflator on A) is itself a frame.*

Proof. By Lemma 1.8 we know that SA is a complete lattice and suprema can be computed pointwise. Thus it will suffice if we show that SA satisfies FDL (the frame distributive law). We show that

$$f \wedge \dot{\bigvee}\mathcal{G} = \dot{\bigvee}\{f \wedge g \mid g \in \mathcal{G}\}$$

for each $f \in SA$ and $\mathcal{G} \subseteq SA$. To do that we simply evaluate both sides at an arbitrary $x \in A$. Thus

$$\begin{aligned}(f \wedge \dot{\bigvee}\mathcal{G})(x) &= f(x) \wedge (\dot{\bigvee}\mathcal{G})(x) \\ &= f(x) \wedge \bigvee\{g(x) \mid g \in \mathcal{G}\} \\ &= \bigvee\{f(x) \wedge g(x) \mid g \in \mathcal{G}\} = (\dot{\bigvee}\{f \wedge g \mid g \in \mathcal{G}\})(x)\end{aligned}$$

to give the required result. The third equality makes use of FDL on A. ■

This result leads to another proof of Lemma 1.13. Consider $f, k \in SA$. Since SA is a frame there is some $l \in SA$ such that

$$f \wedge g \leq k \Longleftrightarrow g \leq l$$

for all $g \in SA$. This l is just the implication $(f \supset k)$ as computed in SA. In general l is not a nucleus, but is when k is a nucleus. We can verify this directly.

2.2 LEMMA. *For each frame A we have*

$$\left.\begin{array}{l} f \in SA \\ k \in NA \end{array}\right\} \Longrightarrow (f \supset k) \in NA$$

(where the implication is computed in SA).

Proof. Let $l = (f \supset k)$, so that l is characterized by the equivalence above. We verify that $f \wedge l^2 \leq k$, and hence $l^2 \leq l$, to give the required result.

For each $x \in A$ we have

$$\begin{array}{rcllcl} (f \wedge l^2)(x) & = & f(x) \wedge l(l(x)) & & & \\ & \leq & f(x) \wedge l(f(x) \wedge l(x)) & & & \\ & \leq & f(x) \wedge l(k(x)) & & & \\ & \leq & f(k(x)) \wedge l(k(x)) & & & \\ & \leq & k^2(x) & & = & k(x) \end{array}$$

to show $l^2 \in \mathcal{G}$, as required. Here the second step holds since $l \in SA$, the third and fifth hold since l is an implication. ■

Observe that the crucial part of this proof is essentially the same calculation as in the proof of Lemma 1.13. In other words, the difference between Lemma 1.13 and this result is more style that content.

This result also gives a variant of Theorem 1.14.

2.3 THEOREM. *For each frame A the assembly NA is a fixed set of the frame SA. In particlar, NA is itself a frame.*

Proof. By Definition 3.17 of [11] and the above Lemma 2.2 it suffices to show that NA is closed under infima as computed in SA, that is pointwise infima. This is just Lemma 1.8. ■

Since NA is a fixed set of SA, then by Lemma 3.18 of [11], it is given by some nucleus on SA. That nucleus is important in its own right and is used, as a closure operation, in situations that have little to do with frames. Here we will describe that nucleus without straying too much into those generalities.

The poset IA of all inflators on the frame A is closed under composition. This allows us to form within IA the finite iterates

$$f^0 = \boldsymbol{id},\ f^1 = f,\ f^2 = f \circ f,\ f^3 = f \circ f \circ f,\ f^4 = f \circ f^3,\ \ldots$$

of an inflator f. Since IA is closed under (non-empty) pointwise suprema we can take these iterates into the transfinite.

2.4 DEFINITION. Let $\mathbb{O}\mathrm{rd}$ be the class of ordinals. For each inflator f on a frame A we set

$$f^0 = \boldsymbol{id} \qquad f^{\alpha+1} = f \circ f^{\alpha} \qquad f^{\lambda} = \dot{\bigvee}\{f^{\alpha} \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. This produces the $\mathbb{O}\mathrm{rd}$-indexed chain

$$(f^{\alpha} \mid \alpha \in \mathbb{O}\mathrm{rd})$$

of ordinal iterates of f. ■

For each frame A and each inflator $f \in IA$, we have an ascending chain

$$\boldsymbol{id} = f^0 \leq f^1 = f \leq f^2 \leq \cdots \leq f^\alpha \leq \cdots \quad (\alpha \in \mathbb{O}\mathrm{rd})$$

through IA indexed by the class $\mathbb{O}$rd of ordinals. On cardinality grounds, this can not increase indefinately. (It can not be longer that the next cardinal $|A|^+$ beyond the cardinality of A.) Thus there is at least one ordinal γ such that $f^{\gamma+1} = f^\gamma$, and then $f^\alpha = f^\gamma$ for each ordinal $\alpha \geq \gamma$. We call the smallest such ordinal the closure ordinal of f, and usually denote it by '∞'. It can happen that ∞ is quite small (even 0,1,or 2), but for appropriate examples it can be arbitrarily large. In some circumstances the size of this closure ordinal contain some information about the complexity of the construction. We will see a hint of this in Section 4, and will look at some aspects in depth elsewhere.

2.5 LEMMA. *For each inflator f on a frame A the closure f^∞ is the least closure operation above f.*

Proof. We know that f^∞ is a closure operation and $f \leq f^\infty$. Conversely, consider any closure operation j with $f \leq j$. We show

$$[\alpha] \quad f^\alpha \leq j$$

by induction on the ordinal α, and then take α to be sufficiently large.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$f \leq j \qquad f^\alpha \leq j$$

by the given property of f and the induction property. Thus

$$f^{\alpha+1} = f \circ f^\alpha \leq j \circ j = j$$

since j is idempotent.

The induction leap to a limit ordinal λ is immediate. ■

An inflator f and its closure f^∞ fix the same elements, that is

$$f(x) = x \Longleftrightarrow f^\infty(x) = x$$

for each $x \in A$. This is another way of moving from f to f^∞. However, this method loses the information encoded in ∞.

Of course, if we start from a special kind of inflator, then we can expect it to have a special kind of closure.

2.6 LEMMA. *Let f be a pre-nucleus on a frame A. Then each iterate f^α is a pre-nucleus, and the closure f^∞ is the least nucleus above f.*

Proof. Assuming that f is a pre-nucleus we show

$$[\alpha] \quad f^\alpha \text{ is a pre-nucleus}$$

by induction on the ordinal α, and then take α to be sufficiently large.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, we remember that

$$f^{\alpha+1} = f \circ f^{\alpha}$$

and that PA is closed under composition.

The induction leap to a limit ordinal λ is slightly more interesting. It suffices to show that

$$f^{\lambda}(x) \wedge f^{\lambda}(y) \leq f^{\lambda}(x \wedge y)$$

for each $x, y \in A$. The converse comparison is immediate since f is an inflator. The definition of f^{λ} and two uses of FDL give

$$f^{\lambda}(x) \wedge f^{\lambda}(y) = \bigvee\{f^{\alpha}(x) \mid \alpha \leq \lambda\} \wedge \bigvee\{f^{\beta}(y) \mid \beta \leq \lambda\} = \bigvee\{f^{\alpha}(x) \wedge f^{\beta}(y) \mid \alpha, \beta \leq \lambda\}$$

so that

$$f^{\lambda}(x) \wedge f^{\lambda}(y) \leq \bigvee\{f^{\gamma}(x) \wedge f^{\gamma}(y) \mid \gamma \leq \lambda\} = \bigvee\{f^{\gamma}(x \wedge y) \mid \gamma \leq \lambda\}$$

as required. The comparison holds since the chain $f^{\bullet}$ is ascending, and the equality holds by the induction hypothesis.

Since each iterate f^{α} is a pre-nucleus, the closure is a pre-nucleus and a closure operation, and hence a nucleus.

Finally consider any nucleus j with $f \leq j$. Then, as in the proof of Lemma 2.5, we have

$$f^{\alpha} \leq j$$

for each ordinal α, and hence $f^{\infty} \leq j$. ■

Using very similar methods we can deal with stable infators with, perhaps, a slightly surprising result.

2.7 LEMMA. *Let f be a stable inflator on a frame A. Then each iterate f^{α} is stable, and each limit iterate f^{λ} is a pre-nucleus. Furthermore, the closure f^{∞} is the least nucleus above f.*

Proof. Assuming that f is a stable inflator we show

$$[\alpha] \quad f^{\alpha} \text{ is stable}$$

by induction on the ordinal α, and then take α to be sufficiently large.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, we remember that

$$f^{\alpha+1} = f \circ f^{\alpha}$$

and that SA is closed under composition.

For the induction leap to a limit ordinal λ consider $x, y \in A$. We have

$$\begin{aligned} f^{\lambda}(x) \wedge y &= \bigvee\{f^{\alpha}(x) \mid \alpha \leq \lambda\} \wedge y \\ &= \bigvee\{f^{\alpha}(x) \wedge y \mid \alpha \leq \lambda\} \\ &\leq \bigvee\{f^{\alpha}(x \wedge y) \mid \alpha \leq \lambda\} \quad \leq \quad f^{\lambda}(x \wedge y) \end{aligned}$$

as required. Here the second step uses FDL, and the third follows by the induction hypothesis.

Using [α] we show that each ordinal iterate f^λ is a pre-nucleus. For $x, y \in A$ we have

$$f^\lambda(x) \wedge f^\lambda(y) = \bigvee\{f^\alpha(x) \wedge f^\beta(y) \mid \alpha, \beta \leq \lambda\}$$

by two uses of FDL. But now a use of [α] and then [β] gives

$$\begin{array}{rcll} f^\lambda(x) \wedge f^\lambda(y) & \leq & \bigvee\{f^\alpha(x \wedge f^\beta(y)) \mid \alpha, \beta \leq \lambda\} & \\ & \leq & \bigvee\{f^\alpha(f^\beta(x \wedge y)) \mid \alpha, \beta \leq \lambda\} & \\ & \leq & \bigvee\{f^\gamma(x \wedge y) \mid \gamma \leq \lambda\} & = \ f^\lambda(x \wedge y) \end{array}$$

as required. Observe how various of the inflator properties are used here.

The fact that f^∞ is the least nucleus above f now follows as in the proof of Lemma 2.5. ■

With these preliminaries it doesn't take too long to prove the following.

2.8 THEOREM. *For a frame A the closure operation $(\cdot)^\infty$ on the frame SA (of all stable inflators on A) is a nucleus.*

Proof. It suffices to show

$$f^\infty \wedge f^\infty \leq (f \wedge g)^\infty$$

for $f, g \in SA$ (since the converse comparison is immediate). To this end let

$$j = f^\infty \qquad k = g^\infty \qquad l = (f \wedge g)^\infty$$

so that $j, k, l \in NA$, and we require $j \wedge k \leq l$.

We first show that

$$[\alpha] \quad f^\alpha \wedge g \leq l$$

by induction on the ordinal α. To help with this notice that

$$[1] \quad f \wedge g \leq l$$

holds (since l is the closure of $f \wedge g$).

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, for each $x \in A$ we have

$$\begin{array}{rcll} (f^{\alpha+1} \wedge g)(x) & = & f(f^\alpha(x)) \wedge g(x) & \\ & \leq & f(f^\alpha(x)) \wedge g(f^\alpha(x)) \wedge g(x) & \\ & \leq & l(f^\alpha(x)) \wedge g(x) & \\ & \leq & l(f^\alpha(x) \wedge g(x)) & \\ & \leq & l^2(x) & = \ l(x) \end{array}$$

as required. Here the third comparison follows by [1] and the fifth by the induction hypothesis [α].

For the induction leap to a limit ordinal λ, for each $x \in A$ we have

$$\begin{aligned}(f^\lambda \wedge g)(x) &= f^\lambda(x) \wedge g(x) \\ &= \bigvee\{f^\alpha(x) \mid \alpha < \lambda\} \wedge g(x) \\ &= \bigvee\{f^\alpha(x) \wedge g(x) \mid \alpha < \lambda\} \ \leq l(x)\end{aligned}$$

as required. Here the third comparsion follows by FDL (on A), and the fourth by the induction hypothesis.

This shows that $[\alpha]$ holds for each ordinal α, and so we have

$$[\infty] \quad j \wedge g \leq l$$

by taking α sufficiently large. But now the same argument (or a different instance of the previous observation) shows that

$$j \wedge g^\alpha \leq l$$

for each ordinal α, and hence

$$j \wedge k \leq l$$

by taking a second sufficiently large ordinal. ■

A symmetric form of the crucial observation of this proof, namely that

$$f \wedge g^\infty \leq (f \wedge g)^\infty$$

for $f, g \in SA$, can be obtained using Lemma 1.13. For $f \in SA$ and $l \in NA$, that lemma shows there is some $m \in NA$ such that

$$f \wedge g \leq l \Longleftrightarrow g \leq m$$

for all $g \in SA$. Now fix $g \in SA$ and let $l = (f \wedge g)^\infty$. We have

$$f \wedge g \leq l$$

so that

$$g \leq m$$

to give

$$g^\infty \leq m$$

(since m is idempotent), and hence

$$f \wedge g^\infty \leq l$$

as required.

We now have a third companion to Theorems 1.14 and 2.3.

2.9 THEOREM. *For each frame A the assembly NA is a quotient frame of the frame SA of all stable inflators on A.*

Proof. By Theorem 2.8 we have a nucleus $(\cdot)^{\infty}$ on SA. The quotient frame

$$(SA)_{\infty}$$

consists of those $f \in SA$ which are idempotent. By Lemma 1.2 these are precisely the nuclei on A. ■

At this stage there is no great merit in preferring any of the proofs of Theorems 1.14, 2.3, 2.9 over the others. Indeed, it could be argued that the three proofs are essentially the same, but presented differently. However, each of the little tricks in the proofs will be useful later, so it does no harm to see them now.

3 Some algebraic properties of the assembly

We have attached to each frame its assembly NA (of all nuclei), and this is itself a frame. Thus A and NA are both objects of the category $\boldsymbol{Frm}$ (of frames and frame morphisms). We will study the structure of NA and its relationship with A. This will take some time, and will go beyond this document. Indeed, there are many questions still unanswered.

Each frame A has a bottom $\perp$ and a top $\top$ which we write as

$$\perp_A \qquad \top_A$$

when it is important to indicate the parent frame. The assembly NA has a bottom and a top

$$\perp_{NA} \qquad \top_{NA}$$

and it is usually wise to distinguish between these extremes and those of A. The bottom and top of NA are the nuclei given by

$$\perp_{NA}(x) = x \qquad \top_{NA}(x) = \top_A$$

for $x \in A$. In particular, $\perp_{NA} = \boldsymbol{id}_A$, the identity function on A. When A is known we sometimes write

$$\perp \ \text{ for } \ \perp_A \qquad \top \ \text{ for } \ \top_A \qquad \perp_N \ \text{ for } \ \perp_{NA} \qquad \top_N \ \text{ for } \ \top_{NA}$$

to avoid unnecessary clutter.

We know that each element a of a frame A gives two nuclei

$$\mathsf{u}_a \qquad \mathsf{v}_a$$

on A where

$$\mathsf{u}_a(x) = a \vee x \qquad \mathsf{v}_a(x) = (a \supset x)$$

for each $x \in A$. In particular, we have

$$\mathsf{u}_{\perp_A} = \boldsymbol{id}_A = \perp_{NA} = \mathsf{v}_{\top_A} \qquad \mathsf{u}_{\top_A} = \top_{NA} = \mathsf{v}_{\perp_A}$$

and we will shortly see a generalization of this relationship.

The two frames A and NA are attached in a canonical way.

3.1 DEFINITION. For a frame A let n_A be the assignment

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ a & \longmapsto & \mathsf{u}_a \end{array}$$

which sends each element $a \in A$ to the trivial nucleus $\mathsf{u}_a \in NA$. ■

Of course, this is not just an assignment.

3.2 LEMMA. *For each frame A the assignment n_A is an injective frame morphism.*

Proof. We need to check that

$$\begin{array}{ll} n_A(\bot_A) \ = \bot_{NA} & \qquad n_A(\top_A) \ \ = \top_{NA} \\ n_A(\bigvee Z) = \bigvee\{n_A(z) \mid z \in Z\} & \qquad n_A(a \wedge b) = n_A(a) \wedge n_A(b) \end{array}$$

that is

$$\begin{array}{ll} \mathsf{u}_{\bot_A} \ = \bot_{NA} & \qquad \mathsf{u}_{\top_A} = \top_{NA} \\ \mathsf{u}_{\bigvee Z} = \bigvee\{\mathsf{u}_z \mid z \in Z\} & \qquad \mathsf{u}_{a\wedge b} = \mathsf{u}_a \wedge \mathsf{u}_b \end{array}$$

for each $a, b \in A$ and $Z \subseteq A$.

The top two are immediate. The bottom right holds since A is distributive and meets in NA are computed pointwise. Only the bottom left need a bit of thought (since suprema in NA need not be computed pointwise).

Let

$$c = \bigvee Z \qquad j = \bigvee\{\mathsf{u}_z \mid z \in Z\}$$

whatever that nucleus j might be. We must show that $j = \mathsf{u}_c$. For each $z \in Z$ we have

$$\mathsf{u}_z(x) = z \vee x \leq c \vee x = \mathsf{u}_c(x)$$

for each $x \in A$, so that $\mathsf{u}_z \leq \mathsf{u}_c$, and hence $j \leq \mathsf{u}_c$. Also

$$\mathsf{u}_c(x) = c \vee x = \bigvee\{z \vee x \mid z \in Z\} \leq j(x)$$

for each $x \in A$, so that $\mathsf{u}_c \leq j$, to give the required result.

This shows that the assignment n_A is a morphism. For $a, b \in A$ we have

$$n_A(a) = n_A(b) \Longrightarrow \mathsf{u}_a = \mathsf{u}_b \Longrightarrow a = \mathsf{u}_a(\bot) = \mathsf{u}_b(\bot) = b$$

to show that n_A is injective. ■

Each frame morphism has a right adjoint, and often this is a useful device. The adjoint of n_A is quite straight forward.

3.3 COROLLARY. *For each frame A the pair of assignments*

$$\begin{array}{ccc} a & \longmapsto & \mathsf{u}_a \\ A & \rightleftarrows & NA \\ j(\bot) & \longleftarrow\!\shortmid & j \end{array}$$

form a frame morphism and its right adjoint.

Proof. By Lemma 3.2 is suffices to observe that

$$a \le j(\bot) \iff \mathsf{u}_a \le j$$

for all $a \in A$ and $j \in NA$. ■

We know that u_a and v_a are the nuclei arising from the canonical morphisms

$$A \longrightarrow [a, \top] \qquad\qquad A \longrightarrow [\bot, a]$$

respectively. This seems to suggest that somehow u_a and v_a have 'complementary' behaviour. We can make that precise.

3.4 THEOREM. *For each frame A and $a \in A$ the two nuclei $\mathsf{u}_a, \mathsf{v}_a$ are complementary in NA, that is*

$$\mathsf{v}_a \vee \mathsf{u}_a = \top_{NA} \qquad \mathsf{v}_a \wedge \mathsf{u}_a = \bot_{NA}$$

hold.

Proof. For the left hand identity let

$$j = \mathsf{v}_a \vee \mathsf{u}_a$$

so we require $j(\bot) = \top$, to give $j = \top_{NA}$. Since

$$\mathsf{v}_a \le j \qquad \mathsf{u}_a \le j$$

we have $a \le j(\bot)$, so that $j(a) \le j(\bot)$. With this the right hand component gives

$$\top = (a \supset a) = \mathsf{v}_a(a) \le j(A) \le j(\bot)$$

for the required result.

For the left hand identity we remember that meets in NA are computed pointwise. Thus for $x \in A$ we have

$$(\mathsf{v}_a \wedge \mathsf{u}_a)(x) = \mathsf{v}_a(x) \wedge \mathsf{u}_a(x) = \mathsf{v}_a(x) \wedge (a \vee x) = (\mathsf{v}_a(x) \wedge a) \vee (\mathsf{v}_a(x) \wedge x) = x$$

since

$$\mathsf{v}_a(x) \wedge a = (a \supset x) \wedge a \le x$$

and $x \le \mathsf{v}_a(x)$. This shows that

$$\mathsf{v}_a \wedge \mathsf{u}_a = \boldsymbol{id}_A = \bot_{NA}$$

as required. ■

All nuclei on A can be built out of the u and v nuclei. To show that we need a bit of a preamble.

Given a pair j, k of nuclei on a frame A, how might we compute the join $j \vee k$? The composite $k \circ j$ is certainly a pre-nucleus but need not be a nucleus. But sometimes it is, and then it's the one we want.

3.5 LEMMA. *For a pair j, k of nuclei on a frame A, if*

$$j \circ k \leq k \circ j$$

then

$$j \vee k = k \circ j$$

holds.

Proof. We know that

$$g = k \circ j$$

is a pre-nucleus. But, using the given comparison, we have

$$g^2 = k \circ j \circ k \circ j \leq k^2 \circ j^2 = k \circ j = g$$

to show that g is already idempotent, and hence is a nucleus. Let $h = j \vee k$. Trivially, we have $j, k \leq g$, so that $h \leq g$. Also

$$g = j \circ k \leq h \circ h = h$$

to show that $g = h$. ■

With this we can generalize the result used in the proof of Theorem 3.4.

3.6 LEMMA. *Let A be a frame A, let $j \in NA$, let $a, b \in A$, and set*

$$l = \mathsf{v}_b \vee j \vee \mathsf{u}_a$$

to obtain a nucleus. Then

$$l = \mathsf{v}_b \circ j \circ \mathsf{u}_a$$

that is

$$l(x) = (b \supset j(a \vee x))$$

for each $x \in A$.

Proof. For an arbitrary nucleus k two simple calculations gives

$$k \circ \mathsf{v}_b \leq \mathsf{v}_b \circ k \qquad \mathsf{u}_a \circ k \leq k \circ \mathsf{u}_a$$

and hence

$$\mathsf{v}_b \vee k = \mathsf{v}_b \circ k \qquad k \vee \mathsf{u}_a = k \circ \mathsf{u}_a$$

by Lemma 3.5. Thus

$$l = \mathsf{v}_b \vee j \vee \mathsf{u}_a = (\mathsf{v}_b \circ j) \vee \mathsf{u}_a = \mathsf{v}_b \circ j \circ \mathsf{u}_a$$

as required. ■

As a particular case of Lemma 3.6 we have

$$\mathsf{v}_a \vee \mathsf{u}_a = \mathsf{v}_a \vee \boldsymbol{id} \vee \mathsf{u}_a = \mathsf{v}_a \circ \boldsymbol{id} \circ \mathsf{u}_a = \mathsf{v}_a \circ \mathsf{u}_a$$

so that

$$(\mathsf{v}_a \vee \mathsf{u}_a)(\bot) = (a \supset a) = \top$$

to show $\mathsf{v}_a \vee \mathsf{u}_a = \top_{NA}$. This is essentially the argument used in the proof of Theorem 3.4. Later, in Section 4, we will look at other methods of computation suprema in NA.

Theorem 3.4 and Lemma 3.6 lead to an important representation of nuclei on a frame A. Given elements $a \leq b$ we say a nucleus j collapses the interval $[a, b]$ if $b \leq j(a)$. Thus j is the least nucleus which collapses all the intervals $[a, j(a)]$.

3.7 LEMMA. *For each frame A and elements a, b we have*

$$\mathsf{u}_b \wedge \mathsf{v}_a \leq j \iff b \leq j(a)$$

for each $j \in NA$. In particular, if $a \leq b$ then $\mathsf{u}_b \wedge \mathsf{v}_a$ is the least nucleus that collapses the interval $[a, b]$.

Proof. Since u_a and v_a are complementary in NA we have

$$\mathsf{u}_b \wedge \mathsf{v}_a \leq j \iff \mathsf{u}_b \leq j \vee \mathsf{u}_a \iff \mathsf{u}_b \leq j \circ \mathsf{u}_a \iff b \leq j(a)$$

as required. The second equivalence follows by Lemma 3.6, and the third follows by evaluation at $\bot$. ■

With this we can show that every nucleus can be generated using these rather simple u and v nuclei.

3.8 THEOREM. *For each frame A we have*

$$j = \bigvee\{\mathsf{u}_{j(a)} \wedge \mathsf{v}_a \mid a \in A\}$$

for each nucleus j on A.

Proof. Let k be the indicated supremum. We do *not* know that k is computed pointwise, and we don't need to.

As a particular case of Lemma 3.7 we have

$$\mathsf{u}_{j(a)} \wedge \mathsf{v}_a \leq j$$

for all $a \in A$, and hence $k \leq j$.

For each $a \in A$ we have

$$\mathsf{u}_{j(a)} \wedge \mathsf{v}_a \leq k$$

so that

$$j(a) \leq k(a)$$

by evaluation at a. This $j \leq k$. ■

This result shows that each assembly NA is generated by its complemented members, in fact, by a particular kind of complemented members. This might suggest that NA has a relatively simple structure. However, as we will find out, this suggestion is misguided.

Here is a simple consequence of Theorem 3.8 which can also lead to incorrect conjectures.

3.9 THEOREM. *The assembly NA of a finite frame A is a boolean algebra.*

Proof. By Theorem 3.8 each $j \in NA$ is a supremum of finitely many complemented members of NA, and hence is itself complemented. ■

We will see in Section 5 that for a finite frame A the assembly NA is just the boolean closure of A, the smallest boolean algebra that includes A. This observation has suggested many conjectures concerning the nature of NA for an arbitrary frame A. All of these are incorrect.

To conclude this section we give another representation of an arbitrary nucleus, but this time in terms of infima and the w nuclei.

We need a simple observation, and it is convenient to put this along side two other similar observation (which, in fact, we have more or less used already).

3.10 LEMMA. *For each frame A we have*

$$\mathsf{u}_a \le j \iff a \le j(\bot) \qquad \mathsf{v}_a \le j \iff j(a) = \top \qquad j \le \mathsf{w}_a \iff j(a) = a$$

for each nucleus $j \in NA$ and element $a \in A$.

Proof. If $\mathsf{u}_a \le j$ then $a \le j(\bot)$ by evaluation at $\bot$. Conversely, if $a \le j(\bot)$ then

$$\mathsf{u}_a(x) = a \vee x \le j(\bot) \vee j(x) = j(x)$$

for each $x \in A$, to give $\mathsf{u}_A \le j$.

If $\mathsf{v}_a \le j$ then $\top \le j(a)$ by evaluation at a. Conversely, suppose $j(a) = \top$ and consider

$$y = \mathsf{v}_a(x) = (a \supset x)$$

for arbitrary $x \in A$. We require $y \le j(x)$. But

$$y \wedge a \le x$$

so that

$$y \le j(y) \wedge j(a) = j(y \wedge a) \le j(x)$$

as required.

If $j \le \mathsf{w}_a$ then $j(a) = a$ by evaluation at a. Conversely, suppose $j(a) = a$ and consider

$$y = j(x) \wedge (x \supset a)$$

for an arbitrary $x \in A$. We have

$$y \le j(x) \qquad x \wedge y \le a$$

so that

$$y = j(x) \wedge y \le j(x) \wedge j(y) = j(x \wedge y) \le j(a) = a$$

to give

$$j(x) \le ((x \supset a) \supset a) = \mathsf{w}_a(x)$$

and hence $j \le \mathsf{w}_a$, since x is arbitrary. ■

The third of these comparisons gives us the following analogue of Theorem 3.8.

3.11 THEOREM. *For each frame A we have*

$$j = \bigwedge\{\mathsf{w}_a \mid a \in A_j\}$$

for each nucleus j on A.

Proof. Given $j \in NA$ let k be the indicated infimum.
We have $j \leq k$ by the third part of Lemma 3.10.
Consider any $x \in A$ and let $a = j(x)$, so that $a \in A_j$ with $k \leq \mathsf{w}_a$ to give

$$k(x) \leq k(a) \leq \mathsf{w}_a(a) = a = j(x)$$

and hence $k \leq j$ since x is arbitrary. ■

We will return to these ideas several times during the course of these and other sets of notes.

4 How to calculate within the assembly

We know that infima in the assembly NA of a frame are computed pointwise. However, suprema and even joins are a different matter.

Consider the two simple nuclei u_a and v_a indexed by the same element a. By Theorem 3.4 we know these are complementary in NA. In particular the join $\mathsf{v}_a \vee \mathsf{u}_a$ is just the top of NA. However, by Example 1.9, the pointwise join $\mathsf{v}_a \dot{\vee} \mathsf{u}_a$ is not even a pre-nucleus (although it is a stable inflator). Notice that

$$(\mathsf{v}_a \dot{\vee} \mathsf{u}_a)(\bot) = \mathsf{v}_a(\bot) \vee \mathsf{u}_a(\bot) = \neg a \vee a$$

which is certainly dense but need not be $\top$. This indicates how the nature of the filter of dense elements of a frame has an impact on the structure of its assembly.

Lemmas 3.5 and 3.6 show that sometimes a join of nuclei seems to be more concerned with composition that with pointwise join. We can take this quite a bit further.

4.1 EXAMPLE. Consider the case of the join $j \vee k$ of two nuclei $j, k \in NA$. Let

$$f = j \circ k \qquad g = k \circ j \qquad h = j \dot{\vee} k$$

to obtain two pre-nuclei f, g and a stable inflator h. We have

$$j, k \leq h \leq f, g \leq h^2$$

by one or two simple calculations. Each of f, g, h has a family of ordinal iterates,

$$f^\alpha \qquad g^\alpha \qquad h^\alpha$$

(for $\alpha \in \mathbb{O}\mathrm{rd}$), and the previous comparisons show that these three chains interlace. In particular, we have

$$f^\infty = g^\infty = h^\infty$$

but where the three closure ordinals can differ by small amounts. This gives us a nucleus above j and k, and a few moment's thought shows that this is nothing more than the join $j \vee k$.

Take another look at Lemma 3.5. From there we see that if $f \leq g$ then g is already a nucleus, and the closure ordinal of f is no more than 2. With $j = \mathsf{u}_a$ and $k = \mathsf{v}_a$ we have $g = \top_{NA}$ and

$$f^2(\bot) = a \vee \neg a$$

with $f(\bot) = \top$, so that the closure ordinal of f can be 2. ■

This gives us a method of computing a join $j \vee k$. We take one of the inflators f, g, h and then iterate. If the closure ordinal is small, then we may have to choose the particular inflator we use with some care. However, if the closure ordinal is large then it usually doesn't matter which inflator we use.

There are several variants on this idea. Here is one I used in [9]. I will not give the proofs here.

4.2 EXAMPLE. Let j be a nucleus and let f be a stable inflator on a frame A, and suppose f^ω is a nucleus. Let

$$g = j \dot{\vee} f \qquad \ell = j \circ f^\omega$$

to produce two stable inflators (and, in fact, ℓ is a pre-nucleus). It is not hard to see that

$$g^\infty = j \vee f^\omega = \ell^\infty$$

but what are the two closure ordinals?

There is a case (concerned with the construction of the reals) where ℓ is already a nucleus but the closure ordinal of g is exactly $\omega + 1$.

There is a different case (concerned with the construction of the irrationals) where the closure ordinal of both g and ℓ is Ω, the first uncountable ordinal.

Presumably there are examples between these two, but I don't have any to hand. ■

Let us now turn to calculation of suprema involving the $\mathsf{w}_\bullet$-nuclei. We know that

$$j \leq \mathsf{w}_a \Longleftrightarrow j(a) = a$$

and

$$j = \bigwedge\{\mathsf{w}_a \mid a \in A_j\}$$

(for $j \in NA$ and $a \in A$). It turns out that the $\mathsf{w}_\bullet$-nuclei are control much of the structure of NA.

4.3 LEMMA. *For each frame A, element $a \in A$, and nucleus $j \in NA$, if $\mathsf{w}_a \leq j$ then*

$$j = \mathsf{w}_a \vee \mathsf{u}_b = \mathsf{w}_a \circ \mathsf{u}_b = \mathsf{w}_b$$

where $b = j(\bot)$.

Proof. We certainly have

$$\mathsf{w}_a \circ \mathsf{u}_b = \mathsf{w}_a \vee \mathsf{u}_b \leq j \leq \mathsf{w}_b$$

so that a comparison

$$\mathsf{w}_b \leq \mathsf{w}_a \vee \mathsf{u}_b$$

will suffice. Equivalently (since u_b and v_b are complementary in the distributive lattice NA) a comparison

$$\mathsf{w}_b \wedge \mathsf{v}_b \leq \mathsf{w}_a$$

will suffice, and this can be verified by evaluation at a.

Since $\mathsf{w}_a \leq j$ we have $a = \mathsf{w}_a(\perp) \leq j(\perp) = b$, and hence $\mathsf{w}_b(a) = b$. Thus

$$(\mathsf{w}_b \wedge \mathsf{v}_b)(a) = b \wedge (b \supset a) \leq a$$

to give the required result. ■

There is another way of seeing at least part of this result.

Remember that for each $a \in A$ the quotient A_{w_a} is a complete boolean algebra, and conversely each boolean quotient of A has this form. Also, and quotient of a complete boolean algebra is boolean and given by a simple nucleus. Now consider $\mathsf{w}_a \leq j$. he quotient A_j of A is a quotient of A_{w_a} and so is boolean. This is a direct proof of the following, but the above proof gives a bit more information.

4.4 COROLLARY. *For a frame the* $\mathsf{w}_\bullet$*-nuclei form an upper section of its assembly.*

What about joins with w_a? A find the way these are computed rather curious.

Consider $a \in A$ and $j \in NA$. How do we calculate $j \vee \mathsf{w}_a$? By Lemma 4.3 we know it is w_b for some $b \in A$. But which b is it? Let

$$f = j \circ \mathsf{w}_a \qquad g = \mathsf{w}_a \circ j \qquad h = \mathsf{w}_a \circ j \circ \mathsf{w}_a$$

so that f, g, h are pre-nuclei and

$$j \vee \mathsf{w}_a = f^\infty = g^\infty = h^\infty$$

is the required join. We show that, in fact, h is already a nucleus.

4.5 LEMMA. *Consider any frame A, element $a \in A$, and nucleus l on A with $\mathsf{u}_a \leq l \leq \mathsf{w}_a$. Then*

$$j \vee \mathsf{w}_a = \mathsf{w}_a \circ j \circ l = \mathsf{w}_b$$

where $b = \mathsf{w}_a(j(a))$.

Proof. Let

$$f = j \circ l \qquad h = \mathsf{w}_a \circ f$$

so that both f and h is a pre-nucleus. On general grounds we have

$$j \vee \mathsf{w}_a = h^\infty$$

so it suffices to show that h is idempotent (and hence is a nucleus).

Observe that

$$\mathsf{w}_a \circ h = h \qquad l \circ h = h$$

since w_a is idempotent and $l \leq \mathsf{w}_a$. As a preliminary step we show that

$$j \circ h = h$$

also holds.

Consider $x \in A$, and let

$$y = f(x) \qquad z = h(x) = \mathsf{w}_a(y)$$

so that, since $l(\bot) = a$, we have

$$j(a) = f(\bot) \leq f(x) \leq y \qquad z \wedge (y \supset a) \leq a$$

for this arbitrary x and related y, z. These give

$$j(z) \wedge (y \supset a) \leq j(z \wedge (y \supset a)) \leq j(a) \leq y$$

so that

$$j(z) \wedge (y \supset a) \leq (y \supset a) \wedge y \leq a$$

and hence

$$z \leq j(z) \leq \mathsf{w}_a(y) = z$$

to show that $j(z) = z$. Thus

$$(j \circ h)(x) = j(z) = z = h(x)$$

to verify the preliminary step.

We now have

$$h^2 = \mathsf{w}_a \circ j \circ l \circ h = \mathsf{w}_a \circ j \circ h = \mathsf{w}_a \circ h = h$$

to show that h is a nucleus, and hence $j \vee \mathsf{w}_a = h$.

Finally, by direct substitution we have

$$h(\bot) = (\mathsf{w}_a \circ j)(a) = b$$

so that $j \vee \mathsf{w}_a = h = \mathsf{w}_b$ by Lemma 4.3. ■

A simple case of this result is worth noting.

4.6 COROLLARY. *For each* $j \in NA$

$$j \vee \mathsf{w}_\bot = \mathsf{w}_b$$

where $b = \neg\neg a$ *for* $a = j(\bot)$.

Continuing with the notation of Lemma 4.5 for the particular case $l = \mathsf{w}_a$ we have

$$f^2 = j \circ h = h$$

and

$$g^2 = h \circ j \leq h \circ h = h$$

so that

$$j \vee \mathsf{w}_a = f^2 = h = g^2$$

is the join. could it be that one or other of f and g is a nucleus. Here is an example to show not.

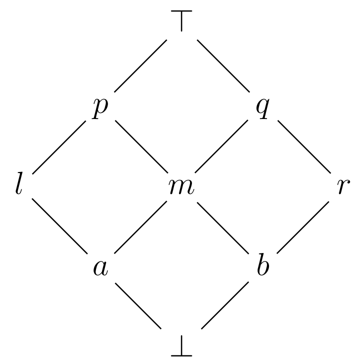
4.7 EXAMPLE. Consider the 9-element frame to the left below. With $j = \mathsf{w}_b$ let

$$f = j \circ \mathsf{w}_a = \mathsf{w}_b \circ \mathsf{w}_a \qquad g = \mathsf{w}_a \circ j = \mathsf{w}_a \circ \mathsf{w}_b \qquad h = \mathsf{w}_a \circ j \circ \mathsf{w}_a = \mathsf{w}_a \circ \mathsf{w}_b \circ \mathsf{w}_a$$

so that

$$\mathsf{w}_a \vee \mathsf{w}_b = f^2 = h = g^2$$

by Lemma 4.5.



			j	f	g	h
x	$x \supset a$	w_a	w_b	$\mathsf{w}_b \circ \mathsf{w}_a$	$\mathsf{w}_a \circ \mathsf{w}_b$	$\mathsf{w}_a \circ \mathsf{w}_b \circ \mathsf{w}_a$
$\top$	a	$\top$	$\top$	$\top$	$\top$	$\top$
q	l	q	$\top$	$\top$	$\top$	$\top$
p	a	$\top$	p	$\top$	$\top$	$\top$
m	l	q	p	$\top$	$\top$	$\top$
r	l	q	r	$\top$	q	$\top$
l	q	l	p	p	$\top$	$\top$
b	l	q	b	$\top$	q	$\top$
a	$\top$	a	p	p	$\top$	$\top$
$\bot$	$\top$	a	b	p	q	$\top$

Consider the table of values to the right above. In column 0, the left hand column, we list the elements x of the frame. In column 1 we list $x \supset a$, and we use this to list $\mathsf{w}_a(x)$ in column 2. By symmetry we can list $j(x)$, and then we can calculate $f(x), g(x), h(x)$ in the final three columns. In particular, we see that f, g, h are distinct. ■

Joins in NA are not as simple as we might think. Given $j, k \in NA$ we may iterate any of $j \circ k, k \circ j, j \dot{\vee} k$ to obtain $j \vee k$. However, the closure ordinal can be arbitrarily high. I will give an illustration of this at the end of this section

What about suprema in NA? Given $\mathcal{J} \subseteq NA$, how might we compute $\bigvee \mathcal{J}$? By Lemma 1.8 the pointwise supremum $\dot{\bigvee} \mathcal{J}$ is a stable inflator which we may close off to a nucleus. A simple argument gives the following.

4.8 LEMMA. *For each family $\mathcal{J}$ of nuclei on a frame A the closure $\left(\dot{\bigvee} \mathcal{J}\right)^{\infty}$ is the supremum of $\mathcal{J}$ in NA.*

There is also a modified version of this method which is sometime useful.

Given $\mathcal{J} \subseteq NA$, let $\mathcal{J}^{\circ}$ be the set of all composites

$$j_1 \circ \cdots \circ j_m$$

for $j_1, \ldots, j_m$. Thus $\mathcal{J}^{\circ}$ is a family of pre-nuclei which is closed under composition. For pre-nuclei f, g we have $f, g \leq g \circ f$, and hence $\mathcal{J}^{\circ}$ is directed. Thus

$$\left(\dot{\bigvee} \mathcal{J}^{\circ}\right)^{\infty}$$

is a pre-nucleus, and it is certainly above each $j \in \mathcal{J}$. Consider any nucleus k with $j \leq k$ for each $j \in \mathcal{J}$. Then

$$j_1 \circ \cdots \circ j_m \leq k^m = k$$

so that

$$\left(\dot{\bigvee}\mathcal{J}^{\circ}\right)^{\infty} \leq k$$

to show that

$$\bigvee \mathcal{J} = \left(\dot{\bigvee}\mathcal{J}^{\circ}\right)^{\infty}$$

holds.

4.9 LEMMA. *For each family $\mathcal{J}$ of nuclei on a frame A the closure $\left(\dot{\bigvee}\mathcal{J}^{\circ}\right)^{\infty}$ is the supremum of $\mathcal{J}$ in NA.*

This method of calculating suprema in NA may look more complicated then the previous method. However, it can be more convenient, when $\mathcal{J}$ is finite and the required closure ordinal is not too large.

Let's now look at another representation result.

We know that

$$j = \bigvee \left\{ \mathsf{u}_{j(a)} \wedge \mathsf{v}_a \mid a \in A \right\}$$

for each nucleus j on a frame A. For some j and a the difference between a and $j(a)$ can be quite large, and hence the component $\mathsf{u}_{j(a)} \wedge \mathsf{v}_a$ can be large. Sometimes we can refine this representation to obtain smaller components.

Suppose we have $j = f^{\infty}$ for some inflator f and suppose the relevant closure ordinal ∞ is quite large. Then, in comparison with the jump from a to $j(a)$, each component jump from x to $f(x)$ is probably quite small. We use this to refine the standard representation.

First of all we look at the behaviour above a given element a.

4.10 CONSTRUCTION. For a given $a \in A$ let

$$a(\alpha) = f^{\alpha}(a)$$

for each ordinal α. Thus

$$a(0) = a \qquad a(\alpha+1) = f(a(\alpha)) \qquad a(\lambda) = \bigvee \{a(\alpha) \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. Similarly set

$$j_{a,0} = \boldsymbol{id} \qquad j_{a,\alpha+1} = \left(\mathsf{u}_{a(\alpha+1)} \wedge \mathsf{v}_{a(\alpha)}\right) \vee j_{a,\alpha} \qquad j_{a,\lambda} = \bigvee \{j_{a,\alpha} \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. ■

For each large ordinal α we have $a(\alpha+1) = a(\alpha) = j(a)$, so that

$$\mathsf{u}_{a(\alpha+1)} \wedge \mathsf{v}_{a(\alpha)} = \boldsymbol{id}$$

and hence $j_{a,\alpha+1} = j_{a,\alpha}$. Thus this ascending chain of nuclei stabilizes at

$$j_a = \bigvee \left\{ \mathsf{u}_{a(\alpha+1)} \wedge \mathsf{v}_{a(\alpha)} \mid \alpha \in \mathbb{O}\mathrm{rd} \right\}$$

where now the size of the components are determined by f-jumps rather than j-jumps.

4.11 LEMMA. *For the chain of nuclei given by Construction 4.10*

$$j_{a,\alpha} = \mathsf{u}_{a(\alpha)} \wedge \mathsf{v}_a$$

holds for each ordinal α.

Proof. We proceed by induction on α.

For the base case, $\alpha = 0$, we have $a(0) = a$ and $j_0 = \boldsymbol{id}$, so the required equality is immediate.

For the induction step, $\alpha \mapsto \alpha + 1$, remembering that $a \leq a(\alpha) \leq a(\alpha + 1)$, we have

$$\begin{aligned}
j_{a,\alpha+1} &= \big(\mathsf{u}_{a(\alpha+1)} \wedge \mathsf{v}_{a(\alpha)}\big) \vee j_{a,\alpha} \\
&= \big(\mathsf{u}_{a(\alpha+1)} \wedge \mathsf{v}_{a(\alpha)}\big) \vee \big(\mathsf{u}_{a(\alpha)} \wedge \mathsf{v}_a\big) \\
&= \big(\mathsf{u}_{a(\alpha+1)} \vee \mathsf{u}_{a(\alpha)}\big) \wedge \big(\mathsf{u}_{a(\alpha+1)} \vee \mathsf{v}_a\big) \\
&\qquad\qquad\qquad\wedge \\
&= \big(\mathsf{v}_{a(\alpha)} \vee \mathsf{u}_{a(\alpha)}\big) \wedge \big(\mathsf{v}_{a(\alpha)} \vee \mathsf{v}_a\big) \\
&= \mathsf{u}_{a(\alpha+1)} \wedge \big(\mathsf{u}_{a(\alpha+1)} \vee \mathsf{v}_a\big) \\
&\qquad\qquad\qquad\wedge \\
&= \top_N \wedge \mathsf{v}_a \qquad\qquad\qquad\qquad = \big(\mathsf{u}_{a(\alpha+1)} \vee \mathsf{v}_a\big)
\end{aligned}$$

as required. The second equality uses the induction hypothesis, and the others various (finitary) distributive laws.

For the induction leap to a limit ordinal λ we have

$$\begin{aligned}
j_{a,\lambda} &= \bigvee \{j_\alpha \mid \alpha < \lambda\} \\
&= \bigvee \{\mathsf{u}_{a(\alpha)} \wedge \mathsf{v}_a \mid \alpha < \lambda\} \\
&= \bigvee \{\mathsf{u}_{a(\alpha)} \mid \alpha < \lambda\} \wedge \mathsf{v}_a \quad = \mathsf{u}_{a(\lambda)} \wedge \mathsf{v}_a
\end{aligned}$$

as required. The second equality uses the induction hypothesis, the second uses the frame distributive law (on NA), and the fourth uses the morphism properties of $\mathsf{u}_\cdot$. ∎

If we now select a sufficiently large ordinal we see that

$$j_a = \mathsf{u}_{j(a)} \wedge \mathsf{v}_a = \bigvee \{\mathsf{u}_{a(\alpha+1)} \wedge \mathsf{v}_{a(\alpha)} \mid \alpha \in \mathbb{O}\mathrm{rd}\}$$

holds. In other words, we have a finer decomposition of each component in the canonical representation of j. Putting all this together gives the following.

4.12 THEOREM. *For each nucleus j and inflator f on A with $j = f^\infty$, the equality*

$$j = \bigvee \{\mathsf{u}_{f(a)} \wedge \mathsf{v}_a \mid a \in A\}$$

holds.

Proof. For each $a \in A$ we have

$$\mathsf{u}_{f(a)} \wedge \mathsf{v}_a \mid a \in A\} \leq \mathsf{u}_{j(a)} \wedge \mathsf{v}_a \mid a \in A\} \leq j$$

to give one comparison. The converse comparison follows from above. ■

Earlier I promised an illustration to show that the calculation of a join in NA may require an arbitrarily long iteration. I have chosen an example which also indicates there is more going on than we might think, at least until we get into particular calculations. Some of the details of this illustration are a bit sketchy, because it requires more background than we have so far. However, it is worth looking at precisely because of the connections it hints at.

This illustration is taken from Section 10.4 of [3].

I will set up the illustration as a series of examples and one result.

4.13 EXAMPLE. Let S be any topological space with topology $\mathcal{O}S$. Initially we assume that S is at least T_1 (and eventually we require S to be T_2). We need a pre-nucleus d on $\mathcal{O}S$ such that

$$d(\{x\}') = S$$

for each $x \in S$. Remember that 'points are closed' since S is T_1. For convenience let us say such a pre-nucleus is deadly because it kills all points.

We also require the closure ordinal of d to be large.

Where might we find such a pre-nucleus?

For each closed set X of S let $\boldsymbol{lim}(X)$ be the set of limit points of X, the set of those points in X that are not isolated in X. In other words, $\boldsymbol{lim}(X)$ is just the Cantor-Bendixson derivative of X.

We easily check that

$$\boldsymbol{lim}(X) \in \mathcal{C}S \qquad \boldsymbol{lim}(X) \subseteq X$$

with

$$Y \subseteq X \Longrightarrow \boldsymbol{lim}(Y) \subseteq \boldsymbol{lim}(X) \qquad \boldsymbol{lim}(X \cup Y) = \boldsymbol{lim}(X) \cup \boldsymbol{lim}(Y)$$

for all $X, Y \in \mathcal{C}S$. Thus, for each $U \in \mathcal{O}S$ we may set

$$\boldsymbol{der}(U) = \boldsymbol{lim}(X')'$$

to obtain a pre-nucleus on $\mathcal{O}S$. Furthermore, we have

$$\boldsymbol{lim}(\{x\}) = \emptyset$$

for each $x \in S$. Thus, by choosing a space S with large CB-rank, we obtain an example of the kind of pre-nucleus we need. ■

It is worth pointing out that an analogue of $\boldsymbol{der}$ can be set up on any frame. This has far reaching consequences for the subject whose ramification are far from being worked out. An investigation of this was begun in [4, 5, 7, 8]. The notes [16] are concerned with this topic.

Using any deadly pre-nucleus we can set up another topology.

4.14 EXAMPLE. Let S be any T_1 and consider any deadly pre-nucleus d on $\mathcal{O}S$. Let

$$S_+ = S_0 + S_1$$

be the disjoint union of two copies of S. Thus we tag each point $x \in S$ in two ways, both as x_0 and as x_1. In the same way, for each $U \in \mathcal{O}S$ we obtain two sets U_0, U_1 where $U_i \subseteq S_i$ for each tag $i \in \{0,1\}$.

Let $\mathcal{O}S_+$ be the collection of all disjoints sums

$$U_0 + V_1$$

where $U, V \in \mathcal{O}S$ with $U \subseteq d(V)$ and $V \subseteq d(U)$. A small calculation shows that $\mathcal{O}S_+$ is a topology on S_+. Only closure under binary intersections is not immediate, and this follows since d is a pre-nucleus. ■

This kind of construction is a simple example of glueing, and can be carried out on any pair of frames with an appropriate pair of maps.

For us the following is the crucial result, which we state without proof.

4.15 LEMMA. *Consider the construction of Example 4.14. If the space S is T_2, then the space S_+ is T_1 and sober.*

Assuming S is T_2, the constructed space S_+ is almost T_2. he only pairs of points that don't have a T_2 separation are those of the form x_0, x_1 arising from the same point $x \in S$. The notion of a sober space is dealt with in [13]. Strictly speaking it isn't needed here but is does stop people asking questions.

With this we can give the example of a large closure ordinal.

4.16 EXAMPLE. Let S be a T_2 space, let d be a deadly pre-nucleus on $\mathcal{O}S$, and let S_+ be the resulting space, as in Example 4.14. The original space S has two copies S_0, S_1 inside S_+, and so produces two spatially induced nuclei

$$[S_0] \qquad [S_1]$$

on the constructed topology. These are given by

$$[S_0](U_0 + V_1) = (S_0 \cup V_1)^\circ = d(V_1) + V_1 \qquad [S_1](U_0 + V_1) = (S_1 \cup U_0)^\circ = U_0 + d(U_0)$$

for $U, V \in \mathcal{O}S$. We look at the joint

$$[S_0] \vee [S_1]$$

of these two nuclei.

Let

$$f = [S_0] \circ [S_1] \qquad g = [S_1] \circ [S_0] \qquad h = [S_0] \dot{\vee} [S_1]$$

so that

$$f(U_0 + V_1) = d^2(U_0 + d(U_0) \qquad g(U_0 + V_1) = d(V_1) + d^2(V_1) \qquad h(U_0 + V_1) = d(V_1) + d(U_0)$$

for $U, V \in \mathcal{O}S$. We also consider k on $\mathcal{O}S_+$ given by

$$k(U_0 + V_1) = d(U_0) + d(V_1)$$

for $U, V \in \mathcal{O}S$. Observe that

$$k, h \leq f, g \leq h^2 = k^2$$

and that

$$f^\infty = g^\infty = h^\infty = k^\infty$$

is the required join. Furthermore, the closure ordinal of each of f, g, h, k is precisely that of d, which can be arbitrarily large. ■

I believe that the ideas of this section can be developed much further. I make a small start on this in [10].

5 The functorial properties of the assembly

We have attached to each frame A a larger frame NA together with an embedding

$$A \xrightarrow{n_A} NA$$

into that assembly. In this section we prove three main results.

- The construction N is an endo-functor on $\boldsymbol{Frm}$.
- The embedding $n_\bullet$ is natural.
- The embedding $n_\bullet$ universally solves a certain problem.

Along the way we gather further information about the assembly. The development in [2], Section 2.8 obtains these results in the order listed. Here we will obtain them in reverse order. As far as I am aware, although it has been known for many years, this development has not appeared in print before.

The crucial property of the embedding is that each element $a \in A$ picks up a complement in NA, since $n_A S(a) = \mathsf{u}_a$ which has a complement v_a in NA. This has several important consequences.

5.1 THEOREM. *For each frame A the injective morphism*

$$A \xrightarrow{n_A} NA$$

is epic.

Proof. Consider a morphism

$$NA \xrightarrow{f} B$$

to an arbitrary frame B. By Theorem 3.4 we have

$$\mathsf{v}_a \wedge \mathsf{u}_a = \bot_{NA} \qquad \mathsf{v}_a \vee \mathsf{u}_a = \top_{NA}$$

so that

$$f(\mathsf{v}_a) \wedge f(\mathsf{u}_a) = \bot_B \qquad f(\mathsf{v}_a) \vee f(\mathsf{u}_a) = \top_B$$

and hence

$$f(\mathsf{v}_a) \qquad f(\mathsf{u}_a)$$

are complementary in B. In particular, $f(\mathsf{v}_a)$ is uniquely determined by

$$f(\mathsf{u}_a) = (f \circ n_A)(a)$$

its complement on B.

Now consider a parallel pair

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

of frame morphism such that

$$f \circ n_A = g \circ n_A$$

that is

$$f(\mathsf{u}_a) = g(\mathsf{u}_a)$$

for each $a \in A$. The observation above shows that

$$f(\mathsf{v}_a) = g(\mathsf{v}_a)$$

for each $a \in A$. Also, for each $j \in NA$, a use of Theorem 3.8 gives

$$\begin{aligned} f(j) &= \bigvee\{f(\mathsf{u}_{j(a)} \wedge \mathsf{v}_a) \mid a \in A\} = \bigvee\{f(\mathsf{u}_{j(a)}) \wedge f(\mathsf{v}_a) \mid a \in A\} \\ g(j) &= \bigvee\{g(\mathsf{u}_{j(a)} \wedge \mathsf{v}_a) \mid a \in A\} = \bigvee\{g(\mathsf{u}_{j(a)}) \wedge g(\mathsf{v}_a) \mid a \in A\} \end{aligned}$$

so that $f(j) = g(j)$, as required. ■

This result shows that

$$A \xrightarrow{n_A} NA$$

is a bimorphism, a morphism that is both monic and epic. However, it is an isomorphism only in the most trivial circumstances.

5.2 THEOREM. *For a frame A the embedding n_A is an isomorphism precisely when A is boolean.*

Proof. Suppose first that A is boolean. Then by Lemma 4.12 of [11] the morphism n_A is surjective, and hence an isomorphism.

Conversely, suppose n_A is an isomorphism. The inverse of n_A is its right adjoint, which is given in Corollary 3.3. Thus the assignment

$$j \longmapsto j(\bot) \longmapsto \mathsf{u}_{j(\bot)}$$

is the identity on NA. In particular

$$j = \mathsf{u}_{j(\bot)}$$

for each $j \in NA$. (This verifies, for this case, the assignment n_A is surjective.)

Now consider any $a \in A$, and let $b = \neg a$. It suffices to show that $b \vee a = \top$. With $j = \mathsf{v}_a$ we have $j(\bot) = b$, so that

$$\mathsf{v}_a = \mathsf{u}_b$$

and hence

$$b \vee a = \mathsf{u}_b(a) = \mathsf{v}_a(a) = \top$$

as required. ■

This shows that the category ***Frm*** is not balanced. It has bimorphisms that are not isomorphisms. In fact, as we will see in [15], the situation is even more complicated. There are frames A with arbitrarily large bimorphic extensions

$$A \longrightarrow B$$

that is the cardinality of B can be arbitrarily large. Furthermore, it seems that for most 'reasonable' spaces the frame $\mathcal{O}S$ has the property.

The embedding $n_A S$ provides complements for all members of the parent frame A, and it does this in a universal way. That phrase need an explanation.

5.3 DEFINITION. We say a frame morphism

$$A \xrightarrow{\ f\ } B$$

solves the complementation problem for A if for each $a \in A$ the element $f(a) \in B$ has a complement in B. ■

For instance, the canonical embedding

$$A \xrightarrow{\ n_A\ } NA$$

solves this complementation problem for A. For each space S the insertion

$$OS \hookrightarrow O^f S$$

solves the complementation problem for OS (in a very crude way).

Notice that if an element a of a frame A (as above) already has a complement a' in A, then $f(a')$ is the complement of $f(a)$ in B. Thus to solve the complementation problem for a frame we need to adjoin complements for all those elements which do not yet have them. Of course, we want the result of this construction to be a frame, and we would like to adjoin the new elements in the free-est possible way. Fortunately, we already know of a frame which does this job without having to go through a messy construction.

5.4 THEOREM. *For each frame A the embedding n_A universally solves the complementation problem for A. That is, for each frame morphism*

$$A \xrightarrow{\ f\ } B$$

which solves the complementation property there is a unique morphism

$$NA \xrightarrow{\ f^\sharp\ } B$$

such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "n_A"'] & & B \\ & NA \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

Proof. For each $a \in A$ we have $n_A(a) = \mathsf{u}_a$ and we know this is complemented in NA. Thus the problem is to show we have a universal solution, that is we have the claimed factorization property.

Consider any morphism

$$\begin{tikzcd} A \arrow[r, shift left, "f = f^*"] & B \arrow[l, shift left, "f_*"] \end{tikzcd}$$

which solves the complementation property. Since n_A is epic, there is at most one possible fill-in morphism $f^\sharp$. Thus it suffices to exhibit an example of such a morphism. We produce this

$$\begin{tikzcd} NA \arrow[r, shift left, "f^\sharp"] & B \arrow[l, shift left, "f_\flat"] \end{tikzcd}$$

together with its right adjoint $f_\flat$, as indicated.

For each $j \in NA$ set

$$f^\sharp(j) = \bigvee\{f(j(x)) \wedge f(x)' \mid x \in A\}$$

where $f(x)'$ is the given complement of $f(x)$ in B. We make a couple of observations.

For $j, k \in NA$

$$j \leq k \Longrightarrow f^\sharp(j) \leq f^\sharp(k)$$

holds. This is a simple consequence of the construction of $f^\sharp$.

For $j, k \in NA$

$$f^\sharp(j) \wedge f^\sharp(k) = f^\sharp(j \wedge k)$$

holds. This is a simple consequence of the construction of $f^\sharp$ and the frame distributive law.

This shows that $f^\sharp$ is a $\{\wedge, \top\}$-morphism. We show that $f^\sharp$ has a right adjoint, and hence is a frame morphism.

To produce this right adjoint $f_\flat$, we use the right adjoint f_* of the given morphism $f = f^*$.

For each $b \in B$ we have a composite frame morphism

$$A \xrightarrow{f^*} B \longrightarrow [b, \top]$$

to a principal upper section of B. Let $f_\flat(b)$ be the kernel of this morphism. Thus

$$y \leq f_\flat(b)(x) \Longleftrightarrow f^*(y) \leq b \vee f(x) \Longleftrightarrow y \leq f_ast(b \vee f^*(x))$$

for each $x, y \in A$. This shows that

$$f_\flat(b)(x) = f_*(b \vee f^*(x))$$

for each $b \in B$ and $x \in A$.

For each $j \in NA$ and $b \in B$ we have

$$\begin{aligned} f^\sharp(j) \le b &\iff (\forall x \in A)[f^*(j(x)) \wedge f^*(x)' \le b] \\ &\iff (\forall x \in A)\,[f^*(j(x)) \le b \vee f^*(x)] \\ &\iff (\forall x \in A)\,[j(x) \le f_*(b \vee f^*(x))] \\ &\iff (\forall x \in A) \quad [j(x) \le f_\flat(b)(x)] \qquad \iff j \le f_\flat(b) \end{aligned}$$

to show $f^\sharp \dashv f_\flat$, as required.

This verifies that $f^\sharp$ is a frame morphism. To complete the proof it remains to show that the triangle of arrows commutes, that is

$$f^\sharp(\mathsf{u}_a) = f(a)$$

holds for each $a \in A$. But, for each $x \in A$ we have

$$f(\mathsf{u}_a(x)) \wedge f(x)' = f(a \vee x) \wedge f(x)' = (f(a) \vee f(x)) \wedge f(x)' = f(a) \vee f(x)'$$

to give

$$f^\sharp(\mathsf{u}_a) = \bigvee\{f(a) \wedge f(x)' \mid x \in A\} = f(a)$$

as required. ■

A simple case of this result is worth looking at.

5.5 COROLLARY. *For each finite distributive lattice A (that is, a finite frame) the embedding n_A is just the boolean closure of A. That is, NA is boolean, and for each lattice morphism*

$$A \xrightarrow{f} B$$

to a boolean algebra B there is a unique morphism

$$NA \xrightarrow{f^\sharp} B$$

such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "n_A"'] & & B \\ & NA \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

Proof. The assembly NA is boolean by Theorem 3.9. The factorization property is given by Theorem 5.4. ■

You should not be taken in by this corollary. The relationship between an arbitrary frame and its assembly is far more complicated that the result suggests. In particular, a frame need not have a boolean reflection, and even when it does the reflection need not be given by the assembly. We will look at this in much more detail in [15]. For now we look at other consequences of Theorem 5.4.

For each frame morphism

$$A \xrightarrow{f} B$$

the composite morphism

$$A \xrightarrow{f} B \xrightarrow{n_B} NB$$

solves the complementation problem for A. Thus an application of Theorem 5.4 leads to the following definition.

5.6 DEFINITION. For each frame morphism

$$A \xrightarrow{f} B$$

let

$$NA \xrightarrow{N(f)} NB$$

be the unique frame morphism such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "n_A"'] & B \arrow[d, "n_B"] \\ NA \arrow[r, "N(f)"'] & NB \end{tikzcd}$$

commutes. ■

This definition with Theorem 5.4 has a couple of immediate consequences. The uniqueness of the factorization together with some standard abstract nonsense gives the following.

5.7 THEOREM. *The two constructions*

$$A \longmapsto NA \qquad\qquad f \longmapsto N(f)$$

form an endo-functor of ***Frm***. *Furthermore, the assignment*

$$Id_{\boldsymbol{Frm}} \xrightarrow{n_\bullet} N$$

is a natural transformation.

With this we have achieved the three main aims of this section (as stated at the beginning). But we are not going to stop here. There is plenty more information we can extract from these constructions.

5.8 LEMMA. *For each frame morphism*

$$A \xrightarrow{f} B$$

we have

$$N(f)(j) = \bigvee\{\mathsf{u}_{f(j(x))} \wedge \mathsf{v}_{f(x)} \mid x \in A\}$$

for each $j \in NA$.

Proof. In the notation of the proof of Theorem 5.4 we have

$$N(f) = (n_B \circ f)^\sharp$$

so that

$$\begin{aligned} N(f)(j) &= \bigvee\{(n_B \circ f)(j(x)) \wedge ((n_b \circ f)(x))' \mid x \in A\} \\ &= \bigvee\{n_B(f(j(x))) \wedge (n_B(f(x)))' \mid x \in A\} \\ &= \bigvee\{\mathsf{u}_{f(j(x))} \wedge (\mathsf{u}_{f(x)})' \mid x \in A\} \qquad = \bigvee\{\mathsf{u}_{f(j(x))} \wedge \mathsf{v}_{f(x)} \mid x \in A\} \end{aligned}$$

to give the required result. For the last step we recall that u_b and v_b are complementary in the target NB. ■

For some special nuclei this formula simplifies.

5.9 LEMMA. *For each frame morphism*

$$A \xrightarrow{f} B$$

both

$$N(f)(\mathsf{u}_a) = \mathsf{u}_{f(a)} \qquad N(f)(\mathsf{v}_a) = \mathsf{v}_{f(a)}$$

hold for each $a \in A$.

Proof. The first of these merely rephrases the naturality of $n_\bullet$. The second holds since $N(f)(\mathsf{v}_a)$ must be the complement of $N(f)(\mathsf{u}_a) = \mathsf{u}_{f(a)}$ in NB. ■

This observation also leads to a different proof of Lemma 5.8.
We start from the canonical representation

$$j = \bigvee\{\mathsf{u}_{j(x)} \wedge \mathsf{v}_x \mid x \in A\}$$

of an arbitrary nucleus on the source A of the frame morphism f. Then, since $N(f)$ is a frame morphism, we have

$$N(f)(j) = \bigvee\{N(f)(\mathsf{u}_{j(x)}) \wedge N(f)(\mathsf{v}_x) \mid x \in A\} = \bigvee\{\mathsf{u}_{f(j(x))} \wedge \mathsf{v}_{f(x)} \mid x \in A\}$$

where the second step follows by Lemma 5.9.
This representation is worthy of further investigation. We will meet it again in Section 6. There are still some simple questions that I can not answer. For instance, I don't know what $N(f)(\mathsf{w}_a)$ is in general.
Lemma 5.9 also ensures that N has some preservation properties.

5.10 LEMMA. *Let*

$$A \xrightarrow{f} B$$

be a frame morphism.

(a) If f is epic, then so is $N(f)$.

(b) If f is surjective, then so is $N(f)$.

Proof. (a) Consider a parallel pair

$$\begin{tikzcd} NB \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

of frame morphisms such that the two composites

$$\begin{tikzcd} NA \arrow[r, "N(f)"] & NB \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

agree. Then all paths from A to C across the diagram

$$\begin{tikzcd} & B \arrow[dr, "n_B"] & & \\ A \arrow[ur, "f"] \arrow[dr, "n_A"'] & & NB \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \\ & NA \arrow[ur, "N(f)"'] & & \end{tikzcd}$$

agree. But f and n_B are epic, so that $g = h$, as required.

(b) It suffices to show that each of the nuclei

$$\mathsf{u}_b \qquad \mathsf{v}_b$$

are in the range of $N(f)$, for $b \in B$. But, since f is surjective, we have $b = f(a)$ for some $a \in A$, and then

$$\mathsf{u}_b = N(f)(\mathsf{u}_a) \qquad \mathsf{v}_b = N(f)(\mathsf{v}_a)$$

as required. ■

Consider a frame morphism $f = f^*$

$$\begin{tikzcd} A \arrow[r, shift left, "f = f^*"] & B \arrow[l, shift left, "f_*"] \end{tikzcd}$$

with its right adjoint f_*, as indicated. This induces a frame morphism $N(f) = N(f)^*$

$$\begin{tikzcd} NA \arrow[r, shift left, "N(f) = N(f)^*"] & B \arrow[l, shift left, "N(f)_*"] \end{tikzcd}$$

between the assemblies with its right adjoin $N(f)_*$, as indicated. We can describe this adjoint.

5.11 THEOREM. *Consider a frame morphism $f = f^* \dashv f_*$, as above. For each $k \in NB$ we have*

$$N(f)_*(k) = f_* \circ k \circ f^*$$

and

$$N(f)(j) \leq k \iff j \leq N(f)_*k \iff f \circ j \leq k \circ f$$

holds for each $j \in NA$.

Proof. We have

$$N(f)_* = (n_B \circ f)_\flat$$

in the notation of the proof on Theorem 5.4. Thus, for each $a, x \in A$, using that definition of $(\cdot)_\flat$ we have [*Have another look at following using 4.2*]

$$\begin{aligned} x \leq N(f)_*(k)(a) &\iff x \leq (n_B \circ f)_\flat(k)(a) \\ &\iff (n_B \circ f)(x) \leq k \vee (n_B \circ f)(a) \\ &\iff \mathsf{u}_{f(x)} \leq k \vee \mathsf{u}_{f(a)} \\ &\iff f(x) \leq k(f(a)) = (k \circ f^*)(a) \quad \iff x \leq (f_* \circ k \circ f^*)(a) \end{aligned}$$

to give the required description of $N(f)_*(k)$.

With this, using the poset adjunction property on two levels we have

$$\begin{aligned} N(f)(j) \leq k &\iff N(f)^*(j) \leq k \\ &\iff j \leq N(f)_*(k) \\ &\iff j \leq f_* \circ k \circ f^* \quad \iff j \circ f \leq k \circ f \end{aligned}$$

as required. ■

Nuclei can be transferred across a frame morphism

$$A \xrightarrow{f} B$$

in either direction. For each $j \in NA$ and $k \in NB$ we know that

$$f \circ j \leq N(f)(j) \circ f \qquad f \circ N(f)_*(k) \leq k \circ f$$

and sometimes this second comparison can be an equality.

5.12 COROLLARY. *Suppose*

$$A \xrightarrow{f} B$$

is a surjective frame morphism. Then

$$f \circ N(f)_*(k) = k \circ f$$

for each $k \in NB$.

Proof. Let $f = f^*$ with right adjoint f_*. Since f is surjective, we have $f^* \circ f_* = \boldsymbol{id}_B$. Thus

$$f \circ N(f)_*(k) = f^* \circ f_* \circ k \circ f^* = k \circ f$$

as required. ■

The result of Theorem 5.11 can be described in terms of commuting diagrams. Consider a frame morphism

$$A \xrightarrow{\ f\ } B$$

and a pair $j \in NA$ and $k \in NB$ of nuclei on the source and target. When can the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_j & B_k \end{tikzcd}$$

be completed to form a commuting square? If there is a missing arrow, then it can only be the restriction of f to A_j composed with $B \longrightarrow B_k$. So the only question is the existence of such an arrow.

Let

$$\begin{tikzcd} A \arrow[r, shift left, "f^*"] & B \arrow[l, shift left, "f_*"] \arrow[r, shift left, "k^*"] & B_l \arrow[l, shift left, "k_*"] \end{tikzcd}$$

be the composite arrow, where each morphism and its right adjoint have been named. In particular $k = k_* \circ k^*$ is the nucleus on B. Notice that the kernel of this composite is

$$f_* \circ k_* \circ k^* \circ f^* = f_* \circ k \circ f^* = N(f)_*(k)$$

by Theorem 5.11. There is a fill-in morphism if and only if j lies below this kernel. In other words, when

$$f \circ j \leq k \circ f$$

holds. In particular, there is such a fill-in for the case $k = N(f)(j)$.

5.13 THEOREM. *Let*

$$A \xrightarrow{\ f\ } B$$

be a frame morphism, let $j \in NA$, and let $k = N(f)(j)$. The resulting square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A_j \arrow[r] & B_k \end{tikzcd}$$

is a push-out.

Proof. Consider a commuting diagram

$$\begin{tikzcd}
& B \arrow[dr, "k^*"] \arrow[drr, "h"] & & \\
A \arrow[ur, "f"] \arrow[dr, "j^*"'] & & B_k & C \\
& A_j \arrow[ur, "f_j"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

where all the arrows have been named and, for convenience, we have given the square a quarter twist. Here $f_j = f|_{A_j}$. We must show there is a unique mediating morphism

$$B_k \xrightarrow{\ m\ } C$$

such that in the diagram

$$\begin{tikzcd}
& B \arrow[dr, "k^*"] \arrow[drr, "h"] & & \\
A \arrow[ur, "f"] \arrow[dr, "j^*"'] & & B_k \arrow[r, "m"] & C \\
& A_j \arrow[ur, "f_j"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

the two flaps commute.

Since k^* is surjective, there can be at most one such morphism m. Thus it suffices to exhibit such a morphism.

To produce m we check that

$$k \leq \ker(h)$$

and then apply the canonical factorization result, Theorem 3.20 of [11]. This will ensure that the h-flap commutes. We then verify directly that the g-flap commutes.

To obtain this comparison we show

$$f \circ j \leq \ker(h) \circ f$$

and then use Theorem 5.11 to get

$$k = N(f)(j) \leq \ker(h)$$

as required.

With $l = \ker(g \circ j^*) = \ker(h \circ f) \in NA$ we have

$$h \circ f \circ l = h \circ f$$

and hence

$$\ker(h) \circ f \circ l = h_* \circ h^* \circ f \circ l = h_* \circ h^* \circ f = \ker(h) \circ f$$

(since $\ker(h) = h_* \circ h^*$ where $h^* = h$). Thus, since $j \le l$ we have

$$f \circ j \le f \circ l \le \ker(h) \circ f \circ l = \ker(h) \circ f$$

as required.

This gives the required arrow m with $h = m \circ k^*$. It remains to check that the g-flap commutes. But

$$g \circ j^* = h \circ f = m \circ l^* \circ f = m \circ f_j \circ j^*$$

and j^* is surjective, so that the required equality follows by cancelling j^*. ■

It is interesting that the account in [2] essentially begins with this push-out property and the derives everything else as a consequence.

6 Certain pushouts in *Frm*

In Section 5 we first obtained various functorial properties of the assembly construction $N(\cdot)$ and then used these to show that various pushouts exist in ***Frm***. In fact, as we show in this section, these pushouts can be described quite directly.

We start from a wedge

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j^*"'] \arrow[dr, phantom, "(*)"] & B \\ A_j & \end{tikzcd}$$

where f is an arbitrary morphism and j is an arbitrary nucleus on A with j^* as the canonical quotient morphism associated with it. (Recall that it is useful to distinguish between a quotient morphism and its controlling nucleus, because of the different way suprema are computed in the source and target.)

Our job is to form the pushout of this wedge.

6.1 DEFINITION. For the wedge $(*)$ let

$$k = \bigvee\{\mathsf{u}_{f(j(x))} \wedge \mathsf{v}_{f(x)} \mid x \in A\}$$

to obtain a nucleus k on B. ■

Observe that by Lemma 5.8 the k is nothing more than $N(f)(j)$. However, we won't use that fact here. There is another explanation of k.

Recall that for each interval $[b, c]$ of B the nucleus

$$\mathsf{u}_c \wedge \mathsf{v}_b$$

is the least one that collapses the interval. More generally, for any family I of intervals of B, the nucleus

$$\bigvee\{\mathsf{u}_c \wedge \mathsf{v}_b \mid [b, c] \in I\}$$

is the least one that collapses all the intervals in I. Thus we have the following.

6.2 LEMMA. *For the wedge* $(*)$, *the constructed nucleus* k *is the least one that collapses all intervals*

$$[f(x), f(j(x))]$$

of B *for arbitrary* $x \in A$.

In other words k is the least nucleus which collapses the transfer of each interval collapsed by j. From this description we can see why k is involved with the pushout construction.

We now have an incomplete square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j^*"'] \arrow[dr, phantom, "(\sqcap)"] & B \arrow[d, "k^*"] \\ A_j & B_k \end{tikzcd}$$

where

$$j^*(x) = j(x) \qquad k^*(y) = k(y)$$

for $x \in A$ and $y \in B$. Our problem is to produce a morphism

$$A \xrightarrow{\ f_j\ } B$$

which makes the square commute. Since j^* is surjective, there can be at most one such morphism. Once we have found it we must show that the resulting square is a pushout.

To produce the morphism f_j we check that

$$j \leq \ker(k^* \circ f)$$

and then use the canonical factorization result, Theorem 3.20 of [11].

6.3 LEMMA. *For the situation described above, the required comparison holds.*

Proof. For $a, x \in A$ we have

$$x \leq \ker(k^* \circ f)(a) \Longleftrightarrow (k^* \circ f)(x) \leq (k^* \circ f)(a) \Longleftrightarrow f(x) \leq k(f(a))$$

so it suffices to check that the right hand comparison holds for $x = j(a)$. But k collapses the interval

$$[f(a), f(j(a))]$$

of B, so that

$$f(j(a)) \leq k(f(j(a))) \leq k(f(a))$$

as required. ■

This shows that the required morphism f_j does exists. Recall also, from Theorem 3.20 of [11], it is given by

$$f_j(x) = (k^* \circ f)(y) = k(f(x))$$

for $x \in A_j$. Thus we do have a commuting square

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j^*"'] \arrow[dr, phantom, "(\Box)"] & B \arrow[d, "k^*"] \\
A_j \arrow[r, "f_j"'] & B_k
\end{tikzcd}
$$

in ***Frm***, and we are ready to prove the required result.

6.4 THEOREM. *For each wedge* $(*)$ *the constructed square* $(\Box)$ *is a pushout.*

Proof. Consider a commuting diagram

$$
\begin{tikzcd}
 & B \arrow[dr, "k^*"] \arrow[drr, "h"] & & \\
A \arrow[ur, "f"] \arrow[dr, "j^*"'] & & B_k & C \\
 & A_j \arrow[ur, "f_j"] \arrow[urr, "g"'] & &
\end{tikzcd}
$$

where, for convenience, we have given the square a quarter twist. We must show there is a unique mediating morphism

$$
B_k \xrightarrow{m} C
$$

such that in the diagram

$$
\begin{tikzcd}
 & B \arrow[dr, "k^*"] \arrow[drr, "h"] & & \\
A \arrow[ur, "f"] \arrow[dr, "j^*"'] & & B_k \arrow[r, "m"] & C \\
 & A_j \arrow[ur, "f_j"] \arrow[urr, "g"'] & &
\end{tikzcd}
$$

the two flaps commute.

Since k^* is surjective, there can be at most one such morphism m. Thus it suffices to exhibit such a morphism.

To produce m we check that

$$
k \le \ker(h)
$$

and then apply the canonical factorization result, Theorem 3.20 of [11]. This will ensure that the h-flap commutes. We then verify directly that the g-flap commutes.

Recall that

$$
y \le \ker(h)(b) \Longleftrightarrow h(y) \le h(b)
$$

for each $b, y \in B$. Now consider the case

$$b = f(a) \qquad y = f(j(a))$$

for $a \in A$. We have

$$\begin{aligned} h(y) &= (h \circ f)(j(a)) \\ &= (g \circ j^*)(j(a)) \\ &= g(j^2(a)) \\ &= g(j(a)) \\ &= (g \circ j^*)(a) \\ &= (h \circ f)(a) \qquad = h(b) \end{aligned}$$

and hence $\ker(h)$ collapse the interval

$$[f(a), f(j(a))]$$

of B. By Lemma 6.2 this gives $k \leq \ker(h)$.

This produces the morphism m which makes the h-flap commute.

Using this and the other commuting cells we have

$$m \circ f_j \circ j^* = m \circ k^* \circ f = h \circ f = g \circ j^*$$

and hence $m \circ f_j = g$ since j_* is surjective. This shows that the g-flap commutes. ■

You have probably noticed a similarity between this proof and that of Theorem 5.13. In fact, the two proofs are essentially the same, and simply present the same facts in a different way.

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[10] H. Simmons: Complemented nuclei on a frame,
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The whole collection of notes can be found on my web pages with

/FRAMES/frames.html

holding the relevant documents. The following are the parts cited in this part.

[11] The basics of frame theory.

[12] The assembly of a frame.

[13] The point space of a frame.

[14] The fundamental traingle of a frame.

[15] Boolean reflections of frames.

[16] The higher level CB properties of frames.

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