First version released on 30 June 2006
This version released on 30 June 2006

The basics of frame theory

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This is the first part of a series of notes [3, 4, 5, 6, 7] on frames, with several more to come. In this part I set down the basic definitions and properties, give a few examples, and begin the discussion of nuclei. The last section looks at various categorical properties, and could be omitted at a first reading. [1] After that you should read [4] and [5] in the order you prefer. That will give you a good grounding in the subject.

The whole collection can be found at [1] If you have never seen this collection before then perhaps you should read [2].

Contents

1 The category of frames

In this section we first set up the definition of a frame and a frame morphism, we then look at two important classes of frames, and finally we sort out some useful gadgetry.

[1] It might make more sense to make Section 5 the bulk of another set of notes on reflections in general for various kinds of structured posets.

1.1 The basic notions

Is there a better way to start than with a definition?

1.1 DEFINITION. A frame is a structure

$$(A,\ \leq,\ \wedge,\ \top,\ \bigvee,\ \perp)$$

where

- $(A, \leq)$ is a complete poset
- $(A, \leq, \wedge, \top)$ is a $\wedge$-semilattice
- $(A, \leq, \bigvee, \perp)$ is a $\bigvee$-semilattice

and where these satisfy

$$\text{(FDL)} \quad a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$$

for each $a \in A$ and $X \subseteq A$. This is the Frame Distributive Law.

A frame morphism

$$A \xrightarrow{\ f\ } B$$

between frames A, B is a function $f : A \longrightarrow B$ which preserves the distinguished attributes.

Let

$$\boldsymbol{Frm}$$

be the category of frames and frame morphisms. ■

We must understand exactly what this definition says so it is worth looking at some of its details.

A frame A is carried by a poset $(A, \leq)$ which is complete, that is both

$$\bigwedge X \qquad \bigvee X$$

exist for each subset $X \subseteq A$. In particular, A has both extremes

$$\bigwedge \emptyset = \top = \bigvee A \qquad \bigwedge A = \perp = \bigvee \emptyset$$

the top and the bottom. Three of these, the two extremes and $\bigvee$ are selected as distinguished attributes. However, $\bigwedge$ is not a distinguished attribute of the frame, but the binary join $\wedge$ is. This signature, the selected attributes, determines the notion of a frame morphism. This is a function, as indicated, which preserves the comparison, in other words it is monotone, and satisfies

$$\begin{array}{cc} f(\top) = \top & f(\perp) = \perp \\ f(x \wedge y) = f(x) \wedge f(y) & f(\bigvee X) = \bigvee f^{\rightarrow}(X) \end{array}$$

for each $x, y \in A$ and $X \subseteq A$ (where A is the source of the morphism). In particular we can have

$$f(\bigwedge X) \neq \bigwedge f^{\rightarrow}(X)$$

for $X \subseteq A$. We will see a general reason for this in Subsection 1.2.

For each function

$$f : A \longrightarrow B$$

we write

$$f^{\rightarrow}(X) = \{f(x) \mid x \in X\} \qquad f^{\leftarrow}(Y) = \{x \in A \mid f(x) \in Y\}$$

for the

direct $\qquad$ inverse

image of

$$X \subseteq A \qquad Y \subseteq B$$

across f, respectively.

As well as carrying these attributes a frame is required to satisfy FDL. As a particular case of this we may take a couple $X = \{x, y\}$ of elements to obtain

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$$

and hence the frame is a distributive lattice. Consequently the frame also satisfies

$$a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$$

for $a, x, y \in A$. However, it can happen that

$$a \vee \bigwedge X \neq \bigwedge \{a \vee x \mid x \in x\}$$

for $a \in A$ and $X \subseteq A$. We will see an example of this in Subsection 1.2.

The import of the FDL can be expressed in a different way. We look at this in Subsection 1.4.

As we have seen, each frame is both a $\wedge$-semilattice and a $\bigvee$-semilattice. More precisely, there is a pair

$$\boldsymbol{Frm} \longrightarrow \boldsymbol{Meet} \qquad \boldsymbol{Frm} \longrightarrow \boldsymbol{Sup}$$

of forgetful functors. We take a closer look at these in Section 5.

The precise definition of ***Meet*** is given in Subsection 5.2, and that of ***Sup*** slightly earlier in Subsection 3.2.

It's time to look at two substantial examples. We make each of these a subsection.

1.2 Topologies

Historically, topologies formed the motivating examples of frames. Before we look at them let's fix some notation which we used throughout these notes, and other sets in the series.

Let S be a topological space. Thus we have two families

$$\mathcal{O}S \qquad \mathcal{C}S$$

of subsets of S, the

$$\text{open} \qquad\qquad \text{closed}$$

subsets of S, respectively. Here $\mathcal{O}S$ *is* the topology. We also have two operations

$$(\cdot)^\circ \qquad\qquad (\cdot)^-$$

on subsets of S, providing the

$$\text{interior} \qquad\qquad \text{closure}$$

of a subset of S, respectively. Any of these four gadgets uniquely determines the topology. We also write

$$(\cdot)'$$

for the set theoretic complementation on S. This, of course, is independent of the particular topology on S.

To be a topology the family $\mathcal{O}S$ must contain the two extremes S and $\emptyset$, and be closed under binary intersections and arbitrary unions. This almost shows that

$$(\mathcal{O}S,\ \subseteq,\ \cap,\ S,\ \bigcup,\ \emptyset)$$

is a frame. All that is needed is the observation that FDL holds since $\mathcal{O}S$ sits inside the power set $\mathcal{P}S$ and the relevant operations are computed set theoretically. This is a useful way if motivating the distinguished attributes of a frame.

We know that for $\mathcal{U} \subseteq \mathcal{O}S$ the intersection $\bigcap\mathcal{U}$ need not be open. The infimum of $\mathcal{U}$ in $\mathcal{O}S$ is given by

$$\bigwedge\mathcal{U} = (\bigcap\mathcal{U})^\circ$$

using the interior operation of the space. Note also that for a topology $\mathcal{O}S$ we need not have

$$W \cup \bigwedge\mathcal{U} = \bigwedge\{W \cup U \mid u \in \mathcal{U}\}$$

for $W \in \mathcal{O}S$ and $\mathcal{U} \subseteq \mathcal{O}S$. Thus the opposite of FDL need not hold in a frame.

Consider a continuous map

$$T \xrightarrow{\ \phi\ } S$$

between a pair spaces S, T. By definition of continuity we have $\phi^{\leftarrow}(U) \in \mathcal{O}T$ for each $U \in \mathcal{O}S$. A couple of simple calculations shows that the assignment

$$OS \xrightarrow{\ \phi^{\leftarrow}\ } OT$$

is a frame morphism. In particular, a frame morphisms need not preserve arbitrary infima.

This construction sets up a contravariant functor

$$\boldsymbol{Top} \xrightarrow{\ \mathcal{O}\ } \boldsymbol{Frm}$$

from spaces to frames. If you are new to this game you should go through the various calculations missing here.

In [5] we show that $\mathcal{O}$ is one half of a contravariant adjunction between ***Frm*** and ***Top***. This connection enables many topological properties to be analysed using frame theoretic methods. That is what point-free topology is about.

1.3 Complete boolean algebras

In this subsection we isolate a 'large' subcategory of ***Frm***.

A boolean algebra is a distributive lattice A for which each element is complemented. That is, for each $a \in A$ there is a , necessarily unique, element $b \in A$ such that

$$a \wedge b = \bot \qquad a \vee b = \top$$

hold. We write $\neg a$ for the element b, so that $\neg(\cdot)$ is a 1-placed operator on A. Notice that

$$\neg\neg a = a$$

(for if b is the complement of a, then a is the complement of b). Notice that any lattice morphism

$$A \xrightarrow{f} B$$

from a boolean algebra A preserves complements, that is if a, b are complementary of A, then $f(a), f(b)$ are complementary in B.

The operation $\neg(\cdot)$ on a boolean algebra A has various properties. In particular we have

$$a \wedge x \leq y \Longleftrightarrow x \leq \neg a \vee y$$

for $a, x, y \in A$. The proof of this is more important than the result. To prove $\Rightarrow$ we remember

$$\neg a \vee a = \top$$

so that

$$x = (\neg a \vee a) \wedge x = (\neg a \wedge x) \vee (a \wedge x) \leq \neg a \vee y$$

using $a \wedge x \leq y$ at the final step. A similar argument starting from

$$\neg a \wedge a = \bot$$

gives the converse implication. This kind of trick is used several times in this subsection.

A complete boolean algebra is a boolean algebra which is complete as a poset. Such an algebra has all the attributes to be a frame, but what about FDL?

1.2 LEMMA. *Each complete boolean algebra is a frame.*

Proof. Let A be a cba. Certainly A is a complete lattice, so it suffices to show that A satisfies the FDL. To this end let

$$l = a \wedge \bigvee X \qquad r = \bigvee\{a \wedge x \mid x \in X\}$$

for $a \in A$ and $x \subseteq A$. The comparison $r \leq l$ is trivial, so it remains to verify the converse. We use the equivalence given above.

For $x \in X$ we have

$$a \wedge x \leq l$$

so that

$$x \leq \neg a \vee l$$

to give

$$\bigvee X \leq \neg a \vee l$$

and hence $r \leq l$, as required. ∎

In any boolean algebra the interaction between meets, joins, and complementation is given by the De Morgan laws. There are similar laws for complete boolean algebras. We need one of these.

1.3 LEMMA. *For an arbitrary subset X of a complete boolean algebra A we have*

$$\bigwedge X = \neg(\bigvee Y)$$

where

$$Y = \{\neg x \mid x \in X\}$$

is the auxiliary subset.

Proof. Let

$$a = \bigwedge X \qquad b = \bigvee Y$$

so that it suffices to show that a, b are complementary, that is

$$a \wedge b = \bot \qquad a \vee b = \top$$

hold.

For each $y = \neg x \in Y$ we have

$$a \wedge y \leq x \wedge \neg x = \bot$$

and hence, since A is a frame,

$$a \wedge b = \bigvee\{a \wedge y \mid y \in Y\} = \bot$$

to give the left hand equality.

Let $c = \neg b$. For each $x \in X$ we have

$$c \wedge \neq x \leq c \wedge b = \bot$$

so that $c \leq x$. Thus $c \leq a$, and hence

$$a \vee b \geq c \vee b = \top$$

to give the right hand equality. ∎

Complete boolean algebras form the objects of two different categories, ***Cba*** and ***CBA***. Given two such algebras A, B and arrow

$$A \xrightarrow{f} B$$

in ***Cba*** is simply a lattice morphism, that is a function that preserves the finitary attributes. As we saw above, such a morphism preserves complements. An arrow in ***CBA***, sometime called a complete morphism, preserves arbitrary infima and suprema, that is

$$f(\bigwedge X) = \bigwedge f^{\rightarrow}(X) \qquad f(\bigvee X) = \bigvee f^{\rightarrow}(X)$$

for each subset X of the source.

Each object of ***CBA*** is a special kind of frame, and each arrow of ***CBA*** is a special kind of frame morphism. Thus we have a forgetful functor

$$\boldsymbol{CBA} \longrightarrow \boldsymbol{Frm}$$

this time with ***Frm*** as the target, not the source.

You might think that there is a third category where the objects are complete boolean algebras, and the arrows are the frame morphisms between these algebras. However, the next result shows that this is just ***CBA***.

1.4 LEMMA. *Let*

$$A \xrightarrow{\quad f \quad} B$$

be a frame morphism from a complete boolean algebra A to a frame B. Then f is a complete morphism, that is

$$f(\bigwedge X) = \bigwedge f^{\rightarrow}(X)$$

for each $X \subseteq A$.

Proof. Let

$$b = f(\bigwedge X) \qquad d = \bigwedge f^{\rightarrow}(X)$$

for the given subset X. The monotonicity of f ensures that $b \leq d$, so it suffices to show the converse comparison.

Let

$$Y = \{\neg x \mid x \in X\} \qquad c = f(\bigvee Y)$$

so that

$$\neg(\bigvee Y) = \bigwedge X$$

(by Lemma 1.3), and hence b and c are complementary in B.

For each $y = \neg x \in Y$ we have

$$d \wedge f(y) \leq f(x) \wedge f(\neg x) = \bot$$

since f passes across $\wedge$. Also f passes across $\bigvee$ so that

$$c = \bigvee\{f(y) \mid y \in Y\}$$

and hence

$$d \wedge c = \bigvee\{d \wedge f(y) \mid y \in Y\} = \bot$$

by the FDL in B. Finally, remembering that

$$c \vee b = \top$$

we have

$$d = d \wedge (c \vee b) = (d \wedge c) \vee (d \wedge b) = d \wedge b \leq b$$

as required. ■

For each pair A, B of complete boolean algebras, we certainly have an inclusions

$$\boldsymbol{CBA}[A, B] \subseteq \boldsymbol{Frm}[A, B]$$

of arrow sets (since each complete morphism is a frame morphism). Lemma 1.4 shows that, in fact, this inclusion is an equality. Thus we have the following.

1.5 THEOREM. *The category* $\boldsymbol{CBA}$ *of complete boolean algebras and complete morphisms is a full subcategory of the category* $\boldsymbol{Frm}$ *of frames.*

In [7] we investigate just how $\boldsymbol{CBA}$ sits inside $\boldsymbol{Frm}$.

1.4 The implication on a frame

A frame is a complete lattice which satisfies the FDL. This equational requirement can be codified in a different way, and produces a useful tool.

1.6 DEFINITION. An implication on a $\wedge$-semilattice A is a two placed operation $(\cdot \supset \cdot)$ such that

$$x \leq (b \supset a) \Longleftrightarrow b \wedge x \leq a$$

for all $a, b, x \in A$.

Trivially, at most one implication can be carried by a $\wedge$-semilattice. Carrying one enforces certain properties.

1.7 LEMMA. *A complete lattice* A *carries an implication precisely when it is a frame.*

Proof. Suppose first that A is a frame. For $a, b \in A$ set

$$(b \supset a) = \bigvee \{x \in A \mid b \wedge x \leq a\}$$

so that

$$b \wedge x \leq a \Longrightarrow x \leq (b \supset a)$$

(for arbitrary $x \in A$). We require the converse. But

$$b \wedge (b \supset a) = \bigvee \{b \wedge x \mid b \wedge x \leq a\} \leq a$$

by the FDL, and this leads to the required result.

Conversely, suppose A carries an implication, and consider any $a \in A$ and $X \subseteq A$. It suffices to show

$$a \wedge \bigvee X \leq \bigvee \{a \wedge x \mid x \in X\}$$

(since the converse comparison is trivial). Let

$$y = \bigvee \{a \wedge x \mid x \in X\}$$

so that

$$a \wedge x \leq y$$

for each $x \in X$. The implication property gives

$$x \leq (a \supset y)$$

for each such x, so that

$$\bigvee X \leq (a \supset y)$$

and hence

$$a \wedge \bigvee X \leq y$$

by a second use of the implication property. ■

This shows that a frame is exactly the same thing as a complete heyting algebra. However, that description can be misleading since frame morphisms need not preserve implication.

It is worth looking at two examples of implication.

1.8 EXAMPLES. (a) For a topology $\mathcal{O}S$ on a space S the implication is given by

$$(V \supset U) = (V' \cup U)^{\circ}$$

for $U, V \in \mathcal{O}S$. To see this consider an arbitrary $W \in \mathcal{O}S$. Then

$$W \subseteq (V' \cup U)^{\circ} \Longleftrightarrow W \subseteq (V' \cup U) \Longleftrightarrow W \cap V \subseteq U$$

to give the required result.

We often use a topology $\mathcal{O}S$ to illustrate various aspects of frame theory. Sometimes we use $\mathcal{O}S$ directly, but sometimes it is more enlightening to look at $\mathcal{C}S$. This often requires the insertion of a complementation at appropriate places. In this particular case, for $X, Y \in \mathcal{C}S$ we have

$$(Y' \supset X')' = (Y \cup X')^{\circ\prime} = (X - Y)^{-}$$

which is clearly important in some situations.

(b) For a complete boolean algebra A the implication is given by

$$(b \supset a) = \neg b \vee a$$

using the negation on A. In other words we have

$$x \leq \neg b \vee a \Longleftrightarrow b \wedge x \leq a$$

for $a, b, x \in A$. This is the observation made just before Lemma 1.2.

We will return to the idea of a negation shortly. ■

There are many identities involving the implication. Some of these are given in the next two results. It isn't worth remembering most of these, but the proof technique – repeated use of the characterizing equivalence – is important.

1.9 LEMMA. *On a frame A we have*

(i) $a \leq (x \supset a)$

(ii) $x \wedge (x \supset a) = x \wedge a$

(iii) $x \leq y \Longrightarrow (y \supset a) \leq (x \supset a)$

(iv) $((\bigvee X) \supset a) = \bigwedge\{(x \supset a) \mid x \in X\}$

for all $a, x, y \in A$ and $X \subseteq A$.

Proof. (i). Since $x \wedge a \leq a$, this is immediate.
(ii). For each $z \in A$ we have

$$\begin{aligned} z \leq x \wedge (x \supset a) &\Longleftrightarrow z \leq x \text{ and } z \leq (x \supset a) \\ &\Longleftrightarrow z \leq x \text{ and } x \wedge z \leq a \qquad \Longleftrightarrow z \leq a \end{aligned}$$

for the required result. The last step here requires just a few moment's thought.
(iii). Let

$$z = (x \supset a)$$

where $x \leq y$. Then

$$x \wedge z \leq y \wedge z \leq a$$

so that

$$z \leq (x \supset a)$$

as required.
(iv). For each $z \in A$ we have

$$\begin{aligned} z \leq (\bigvee X) \supset a &\Longleftrightarrow z \wedge \bigvee X \leq a \\ &\Longleftrightarrow \bigvee\{z \wedge x \mid \in X\} \leq a \\ &\Longleftrightarrow (\forall x \in X)[z \wedge x \leq a] \\ &\Longleftrightarrow (\forall x \in X)[z \leq (x \in a)] \quad \Longleftrightarrow z \leq \bigwedge\{(x \supset a) \mid x \in X\} \end{aligned}$$

for the required result. notice the use of FDL at the second step. ■

Part (iv) of this result is one of the few identities worth remembering. Notice that by taking $X = \{x, y\}$ we obtain

$$((x \vee y) \supset a) = (x \supset a) \wedge (y \supset a)$$

as a particular case.
The next definition may look a little strange but, as we will learn, the operator produced is very important.

1.10 DEFINITION. For a frame A we set

$$\mathsf{w}_a(x) = ((x \supset a) \supset a)$$

for each $a, x \in A$ to produce an operator w_a on A. ■

We will use these operators quite a lot. Here are their simple properties.

1.11 LEMMA. *For each element a of a frame A, the operator w_a is inflationary, monotone, and idempotent, and satisfies*

$$(\mathsf{w}_a(x) \supset a) = \mathsf{w}_a(x \supset a) = (x \supset a) \qquad \mathsf{w}_a(x \wedge y) = \mathsf{w}_a(x) \wedge \mathsf{w}_a(y)$$

for each $x, y \in A$.

Proof. For each $x \in A$ a use of Lemma 1.9(ii) gives

$$x \wedge (x \supset a) = x \wedge a \leq a$$

so that

$$x \leq ((x \supset a) \supset a) = \mathsf{w}_a(x)$$

to show that w_a is inflationary.

Two uses of Lemma 1.9(iii) shows that w_a is monotone.

Before we show that w_a we verify the left hand identity. To this end let

$$z = (\mathsf{w}_a(x) \supset a) = \mathsf{w}_a(x \supset a)$$

where the right hand equality is an immediate consequence of the definition of w_a. Since $x \leq \mathsf{w}_a(x)$, we have

$$x \wedge z \leq \mathsf{w}_a(x) \wedge z \leq a$$

so that

$$z \leq (x \supset a) \leq \mathsf{w}_a)x \supset a) = z$$

for the required result.

With this we have

$$\mathsf{w}_a^2(x) = ((\mathsf{w}_a(x) \supset a) \supset a) = ((x \supset a) \supset a) = \mathsf{w}_a(x)$$

to show that w_a is idempotent.

Finally, for $x, y \in A$ we have

$$\mathsf{w}_a(x \wedge y) \leq \mathsf{w}_a(x) \leq \mathsf{w}_a(y)$$

since w_a is monotone. This a converse comparison will give the remaining required result. To this end let

$$z = \mathsf{w}_a(x) \leq \mathsf{w}_a(y)$$

so that

$$z \leq \mathsf{w}_a(x) \qquad z \leq \mathsf{w}_a(y)$$

to give

$$z \wedge (x \supset a) \leq a \qquad z \wedge (y \supset a) \leq a$$

and hence, using Lemma 1.9(iv), we have

$$z \wedge ((x \wedge y) \supset a) = z \wedge ((x \supset a) \vee (y \supset a)) = (z \wedge (x \supset a)) \vee (z \wedge (y \supset a)) \leq a$$

which leads to the required comparison. ■

A particular case of implication gives the negation

$$\neg a = (a \supset \bot)$$

of an element $a \in A$. This is characterized by

$$z \leq \neg a \Longleftrightarrow a \wedge z = \bot$$

for $z \in A$. Now this terminology and notation may be confusing you, for in Subsection 1.3 we have used both in conjunction with a complete boolean algebra. Let's clear that up.

We say an element $a \in A$ of a frame A is complemented if

$$a \wedge b \qquad a \vee b = \top$$

for some (necessarily unique) element $b \in A$. We then say that b is the complement of a in A. Notice that this terminology agrees with the boolean case.

1.12 LEMMA. *Let A be an arbitrary frame, and consider $a \in A$.*
The element a is complemented precisely when $\neg a \vee a = \top$.
Furthermore, if a is complemented, then its complement is its negation $\neg a$.

Proof. Suppose first that a is complemented, that is

$$a \wedge b \qquad a \vee b = \top$$

for some $b \in A$. The first of these gives

$$b \leq \neg a$$

and the second gives

$$\neg a = \neg a \wedge (a \vee b) = (\neg a \wedge a) \vee (\neg a \wedge b) = (\neg a \wedge b) \leq b$$

to show that $\neg a = b$, and hence

$$\neg a \vee a = b \vee a = \top$$

holds.

Conversely, suppose

$$\neg a \vee a = \top$$

then, with $b = \neg a$, we have

$$a \wedge b = a \wedge \neg a = \bot \qquad a \vee b = a \vee \neg a = \top$$

to show that a is complemented. ■

This result show that we can define the notion of a complement element of a frame in terms of the behaviour of the negation operation on that frame. We extend this idea.

1.13 DEFINITION. An element $a \in A$ of a frame A is, respectively,

complemented regular dense

if

$$a \vee \neg a \qquad a = \neg\neg a \qquad \neg a = \bot$$

using the negation operation on A. ■

Notice that double negation $\neg\neg(\cdot)$ is nothing more than the operation $\mathsf{w}_\bot$ of Definition 1.10. In particular, for each element a we have

$$a \leq \neg\neg a \qquad \neg\neg\neg a = \neg a$$

by a particular case of a part of Lemma 1.11. This shows that a is dense precisely when $\neg\neg a = \top$.

This observation also shows that $\neg\neg a$ is regular (no matter what element a we start with). Also, as in the proof of Lemma 1.11 we have

$$\neg(a \vee \neg b) = \neg a \wedge \neg b$$

for all elements a, b. In particular, and element of the form $a \vee \neg a$ is always regular. Finally, since

$$a = \neg\neg a \wedge (a \vee \neg a)$$

we see that each element of a frame is the meet of a regular element and a denser element. This is a standard lattice theoretic observation.

Definition 1.13 uses standard terminology from lattice theory, but the words are also used in topology. There is no conflict.

1.14 EXAMPLE. Consider a topology $\mathcal{O}S$ viewed as a frame. For each $U, V \in \mathcal{O}S$ we have

$$V \cap U = \emptyset \iff V \subseteq U' \iff V \subseteq U'^{\circ} = U^{-\prime}$$

to show that $U^{-\prime}$ is the negation of U in $\mathcal{O}S$. In particular, U is complemented precisely when

$$U \cup U^{-\prime} = S \iff U^{-\prime} = U' \iff U^{-} = U$$

in other words when U is clopen.[2]

Since

$$\neg\neg U = U^{-\prime-\prime} = U^{-\circ}$$

we see that U is regular in the sense of Definition 1.13 precisely when it is topologically regular.

Finally, U is dense in the sense of Definition 1.13 precisely when

$$U^{-\circ} = \neg\neg U = S$$

equivalently when

$$U^{-} = S$$

that is when U is topologically dense. ■

[2] I sometimes tell students that this is where the negation symbol comes from. Some of them don't believe me.

In time we will develop the ideas in this example quite a bit further.

Every complemented element is regular, but in general the converse doesn't hold. In fact, we have the following.

1.15 LEMMA. *A frame A is boolean precisely when each element is regular.*

Proof. Suppose A is boolean, so that

$$a \vee \neg a = \top$$

for each element A. On replacing a be $\neg a$ we have

$$\neg a \vee \neg\neg a = \top$$

so that $\neg\neg a$ is the complement of $\neg a$. But this complement is a, so that $\neg\neg a = a$.

Conversely, suppose that $\neg\neg(\cdot)$ is the identity function on A. For each $a \in A$ we have

$$\neg a \wedge \neg\neg a) = \bot$$

so that

$$a \vee \neg a = \neg\neg(a \vee \neg a) = \neg(\neg a \wedge \neg\neg a) = \neg\bot = \top$$

to show that $\neg a$ is the complement of a. ■

Most of the calculations in this section have been rather standard, only only occasionally have we needed the completeness of the frame. In due course we will need to go through several calculations which are more frame specific.

1.5 Morphisms as poset adjunctions

Each frame morphism is a $\bigvee$-preserving function between complete posets. In particular, as a poset map this function has a right adjoint. This is a useful gadget, and is worth a bit of notation.

1.16 DEFINITION. For each frame morphism $f = f^*$ from frame A to frame B

$$\begin{tikzcd} A \arrow[r, shift left, "f^*"] & B \arrow[l, shift left, "f_*"] \end{tikzcd}$$

the right adjoint f_* is the unique monotone map from B to A such that

$$f^*(a) \leq b \Longleftrightarrow a \leq f_*(b)$$

for all $a \in A$ and $b \in B$. ■

Each of A and B is a poset. If we view each as a category that a functor from one to the other is just a monotone map. Two such maps are adjoint as functors precisely when the are adjoint in the sense of Definition 1.16. Here we use upper and lower decorations

$$f^* \dashv f_*$$

to distinguish between the left and the right component of the adjunction. (However, you are warned that some writers don't use this convention, perhaps because they don't understand it.) Finally, not that not every poset adjunction $f^* \dashv f_*$ between a pair of frames gives a frame morphism f^*. This left adjoint must also preserve finitary meets.

A simple exercise show that for each frame morphism $f^* \dashv f_*$ (or any poset adjunction) we have

$$f_*(\bigwedge Y) = \bigwedge f_*^{\rightarrow}(Y)$$

for each $Y \subseteq B$. However, f_* need not be a frame morphism. In fact, it need not preserve even binary joins.

Each continuous map

$$T \xrightarrow{\ \phi\ } S$$

between a pair of spaces gives a frame morphism

$$\mathcal{O}S \xrightarrow{\ \phi^{\leftarrow}\ } \mathcal{O}T$$

between the carried topologies, and hence produces an adjoint pair

$$\begin{tikzcd} \mathcal{O}S \arrow[r, shift left, "\phi^*"] & \mathcal{O}T \arrow[l, shift left, "\phi_*"] \end{tikzcd}$$

where $\phi^* = \phi^{\leftarrow}$ is the left adjoint. What is the right adjoint ϕ_*? It is reasonable to expect this to have something to do with direct images.

1.17 LEMMA. *For each continuous map ϕ, as above, the right adjoint ϕ_* is give by*

$$\phi_*(W) = \phi^{\rightarrow}(W')^{-\prime}$$

for each $W \in \mathcal{O}T$.

Proof. Consider $U \in \mathcal{O}S$ and $W \in \mathcal{O}T$. We have

$$\begin{aligned} U \subseteq \phi^{\rightarrow}(W')^{-\prime} &\iff \phi^{\rightarrow}(W')^{-} \subseteq U' \\ &\iff \phi^{\rightarrow}(W') \subseteq U' \\ &\iff (\forall t \in T)[t \in W' \Longrightarrow \phi(t) \in U'] \\ &\iff (\forall t \in T)[\phi(t) \in U \Longrightarrow t \in W] \\ &\iff (\forall t \in T)[t \in \phi^{\leftarrow}(U) \Longrightarrow t \in W] \quad \iff \phi^{\leftarrow}(U) \subseteq W \end{aligned}$$

to verify the required equivalence. ■

We said earlier that the right adjoint f_* of a frame morphism need not preserve even binary joins. We can illustrate this using two very simple spaces.

1.18 EXAMPLE. Consider the 2-point spaces

$$T = \{l, r\} \qquad S = \{0, 1\}$$

with T discrete and with

$$\mathcal{O}S = \{\emptyset, \{1\}, S\}$$

(so that S is sierpinski space). Let

$$\phi(l) = 0 \qquad \phi(r) = 1$$

to produce a continuous map from T to S. With

$$U = \{l\} \qquad V = \{r\}$$

we have

$$\begin{array}{lllll}
\phi_*(U \cup V) = \phi_*(T) & = \phi^{\rightarrow}(\emptyset)^{-\prime} = \emptyset^{-\prime} & = \emptyset' & = S \\
\phi_*(U) & = \phi^{\rightarrow}(U')^{-\prime} = \phi(r)^{-\prime} & = \{1\}^{-\prime} = S' & = \emptyset \\
\phi_*(V) & = \phi^{\rightarrow}(V')^{-\prime} = \phi(l)^{-\prime} & = \{0\}^{-\prime} = \{0\} = \{1\}
\end{array}$$

so that

$$\phi_*(U \cup V) = S \qquad \phi_*(U) \cup \phi_*(V) = \{1\}$$

to show that ϕ_* is not a $\cup$-morphism. ■

We will, of course, have a lot more to say about frame morphisms in general.

2 The universal algebra of frames

Here by 'universal algebra' I don't mean anything very sophisticated; just some information about subframes, and a few of the simple categorical properties of ***Frm***. Quotient frame are left until the next section.

Given a frame $A = (A, \leq, \wedge, \top, \bigvee, \bot)$ a subframe is a subset $B \subseteq A$ which is itself a frame under the restriction of the distinguished attributes of A. Thus $\top, \bot \in B$ and B is closed under binary meets and arbitrary suprema. The FDL then automatically transfers to B.

This is as must as we need to know about subframes, for they are not very interesting. However, let's not dismiss them just yet.

Given frames $B \subseteq A$ we know that any computation involving $\wedge$ and $\bigvee$ done in B agrees with the corresponding computation done in A. However, the implication and negation on B need not agree with those on A.

2.1 EXAMPLE. Consider two topologies

$$\mathcal{O}S \subseteq \square S$$

on the same set S. Let us write

$$(\cdot)^{\circ} \qquad (\cdot)^{\square}$$

for the two respective interior operations. Thus

$$E^{\circ} \subseteq E^{\square}$$

for each subset $E \subseteq S$, and these can be far apart. For $U, V \in \mathcal{O}S$ the two implications are given by

$$(V \supset U) = (V' \cup U)^{\circ} \qquad (V \sqsupset U) = (V' \cup U)^{\square}$$

respectively, and these can be very different. ■

Recall that, respectively, an arrow

$$B \xrightarrow{\quad m \quad} A \qquad\qquad A \xrightarrow{\quad e \quad} B$$

of $\boldsymbol{Frm}$ is

monic epic

if for each parallel pair arrows

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

the implication

$$m \circ f = m \circ g \Longrightarrow f = g \qquad\qquad f \circ e = g \circ e \Longrightarrow f = g$$

hold. Since the arrows in $\boldsymbol{Frm}$ are functions (of a certain kind) we have

$$m \text{ injective } \Longrightarrow m \text{ monic} \qquad\qquad e \text{ surjective } \Longrightarrow m \text{ epic}$$

by the obvious cancellation argument.

We show that the monics of $\boldsymbol{Frm}$ are precisely the injective arrows, but there are epics which are not surjective.

To deal with monics we use the 3-element frame

$$\mathsf{3} = \begin{pmatrix} 1 \\ \star \\ 0 \end{pmatrix}$$

where $0 < \star < 1$. This can be used to separate the elements of an arbitrary frame.

Consider any frame B and any element $b \in B$. consider the function f as on the left

$$f : \mathsf{3} \longrightarrow B \qquad \begin{array}{l} f(1) = \top \\ f(\star) = b \\ f(0) = \bot \end{array}$$

given by the equalities on the right. Almost trivially this is a frame morphism. Trivially, every frame morphism from $\mathsf{3}$ arises in this way.

2.2 THEOREM. *In the category* $\boldsymbol{Frm}$ *the monics are precisely the injective morphisms.*

Proof. On general grounds each injective morphism is monic.

Conversely, suppose

$$B \xrightarrow{\quad m \quad} A$$

is monic and consider any $b, c \in B$ with $m(b) = m(c)$. We require $b = c$. Let

$$\begin{tikzcd} \mathsf{3} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

be the pair of arrows determined by

$$f(\star) = b \qquad g(\star) = c$$

with $f(0), f(1), g(0), g(1)$ as they must be. We have

$$(m \circ f)(\star) = m(b) = m(c) = (m \circ g)(\star)$$

so that

$$m \circ f = m \circ g$$

and hence

$$f = g$$

since m is monic. This gives

$$b = f(\star) = g(\star) = c$$

as required. ■

Not every epic in ***Frm*** is surjective. We use a particular example to show this. In fact, the example illustrates a little bit more.

Remember that a bimorphism is a morphism that is both monic and epic. Thus each isomorphism is a bimorphism. we give an example of a bimorphism that is not surjective, and not an isomorphism.

2.3 EXAMPLE. Let S be a T_1 topological space, with topology $\mathcal{O}S$. The insertion

$$\mathcal{O}S \hookrightarrow \mathcal{P}S$$

is certainly monic, and is surjective only when S is discrete (so that $\mathcal{O}S = \mathcal{P}S$). We show that the insertion is epic.

Since S is T_1, we know that 'points are closed'. For each point $p \in S$ let

$$U_p = \{p\}' \qquad X_p = \{p\}$$

to obtain the

$$\text{open} \qquad \text{closed}$$

set attached to p. We have

$$E = \bigcup \{X_p \mid p \in E\}$$

for each subset $E \subseteq S$.

Consider any morphism

$$\mathcal{P}S \xrightarrow{f} C$$

to an arbitrary frame C. Since U_p and X_p are complements in $\mathcal{P}S$, the values

$$f(U_p) \qquad f(X_p)$$

are complements in C. Thus the value $f(X_p)$ is uniquely determined by $f(U_p)$.

consider two morphism

$$\begin{tikzcd} \mathcal{P}S \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd}$$

which agree on $\mathcal{O}S$. We show that $f = g$, and hence the insertion is epic.

For each $p \in S$ we have

$$f(U_p) = g(U_p)$$

and hence

$$f(X_p) = g(X_p)$$

by the remarks above. Consider any $E \subseteq S$. Using the representation of E in terms of the X_p, the preservation properties of f and g give

$$f(E) = \bigvee\{f(X_p) \mid p \in X\} = \bigvee\{g(X_p) \mid p \in X\} = g(E)$$

for the required result. ■

To conclude this section we make a simple observation which, as the story unfolds, will become more and more important.

Let

$$\mathsf{2} = \binom{1}{0} \qquad \mathsf{1} = (\star)$$

be the 2-element and 1-element frame, respectively. Almost trivially, these are the

initial final

objects of ***Frm***, respectively. In other words, for each frame A there are unique morphisms

$$\mathsf{2} \longrightarrow A \qquad\qquad A \longrightarrow \mathsf{1}$$

given in the obvious way.

This uses $\mathsf{2}$ as a frame. We can also view it as a topological space. We don't use the discrete topology, rather we use

$$\mathcal{O}\mathsf{2} = \{\emptyset, \{\star\}, \mathsf{2}\}$$

the upper section topology. Notice that $\mathsf{3} \cong \mathcal{O}\mathsf{2}$.

When viewed in this way we refer to the space $\mathsf{2}$ as the sierpinski space. As we will find out, this has a controlling interest in point-free topology.

3 Quotients of frames

A quotient of a frame A is a surjective morphism

$$A \xrightarrow{f} B$$

to some frame B. In particular, the structure of the target B is completely determined by the structure of the source A and the nature of f. As in almost any algebraic situation, the structure of B can be coded by a congruence on A. However, frames being what they are, this congruence on A can be replaced by another far more useful and amenable gadget.

These gadgets, the nuclei on A, are the central components in a family of techniques which are a distinctive feature of the analysis of frames. We begin to develop these in Section 4, then more extensively in [4]. In this section we concentrate on characterizing quotients and thereby showing how nuclei first appear. To do that (and to prepare for more sophisticated uses) we set the development in a broader context. Thus we look at quotient in the category ***Sup*** of $\bigvee$-semilattices, and to do that we look briefly at quotients in the category ***Set*** of sets.

3.1 Quotients in *Set*

Almost all algebraic quotients can be obtained by refining a simple construction in the category ***Set*** of sets and functons.

Let A be a set, and let $\sim$ be an equivalence relation on A. Let

$$A/\!\sim$$

be the set of blocks of $\sim$, the set of $\sim$-equivalence classes, and let

$$A \xrightarrow{\ \eta\ } A/\!\sim$$

be the canonical surjection. Thus for each $a \in A$

$$\eta(a) = \{x \in A \mid a \sim x\}$$

is the block to which a belongs. In particular, we have

$$\eta(x) = \eta(y) \Longleftrightarrow x \sim y$$

for $x, y \in A$.

Conversely consider any function

$$A \xrightarrow{\ f\ } B$$

from A. Setting

$$x \approx y \Longleftrightarrow f(x) = f(y)$$

for $x, y \in A$ produces an equivalence relation on A. Because of what comes later it is useful to think of this as the kernel of f.

These two construction are related by the following, rather simple but fundamental, result.

3.1 THEOREM. *Let $\sim$ be an equivalence relation on the set A and let f be a function from A, as above. Suppose $\sim$ is included in the kernel of f, that is we have*

$$x \sim y \Longrightarrow f(x) = f(y)$$

for $x, y \in A$. Then there is a unique function $f^\sharp$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\eta"'] & & B \\ & A/{\sim} \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

Proof. Consider any block $\alpha \in A/{\sim}$ and set

$$f^\sharp(\alpha) = f(a)$$

for any $a \in \alpha$. The compatibility of $\sim$ with f ensures this is a well-defined function

$$A/{\sim} \xrightarrow{f^\sharp} B$$

and trivially we have

$$(f \circ \eta)(a) = f(a)$$

for each $a \in A$. This shows there is at least one function that makes the triangle commutes. Since η is surjective this is the only possible fill-in function. ■

There are several algebraic refinements of this result. When the two sets A, B carry similar algebraic structures and f is a companion morphism we may imposed on $A/{\sim}$ a similar algebraic structure such that both η and $f^\sharp$ become morphisms.

3.2 Quotients in ***Sup***

The construction of Theorem 3.1 can be refined to obtain a similar factorization of morphisms in many algebraic situations. In the first instance we replace the arbitrary equivalence relation $\sim$ by a congruence, a special kind of equivalence relation that respects the distinguished attributes. Then, if we are lucky, we can replace that congruence by another, more amenable, gadget. For instance, for groups or rings the congruence can be replaced by a particular one of its blocks, the normal subgroups or ideals, respectively. However, these two version still involve dealing with block representatives, which is always a nuisance. A similar thing happens with $\bigvee$-semilattices and frames, but for these the replacement has a quite different character and does not require block representatives.

Before we get to these new gadgets we nee to sort out what a ***Sup***-congruence is. And before that I suppose I should tell you what the category $\bigvee$ is.

3.2 DEFINITION. The objects of the category ***Sup*** are the complete posets. When viewed in this way we refer to such an object as a $\bigvee$-semilattice.

An arrow

$$A \xrightarrow{f} B$$

of ***Sup***, or $\bigvee$-morphism, is a function

$$f : A \longrightarrow B$$

between to $\bigvee$-semilattices such that

$$f(\bigvee X) = \bigvee f^{\rightarrow}(X)$$

for each $X \subseteq A$. ■

Each $\bigvee$-semilattice is a complete poset, and so has all infima as well as all suprema. However, a $\bigvee$-morphism f need not preserve any infima, even binary meets. Note that since $\perp = \bigvee \emptyset$, such a $\bigvee$-morphism must preserve bottom, but it need not preserve top. You should think about this, and make sure you understand why.

We can now begin to sort out what a ***Sup***-congruence is. To do that it is convenient to introduce a bit of temporary notation.

Let A be a $\bigvee$-semilattice, and $\sim$ be an equivalence relation on A. When is $\sim$ a ***Sup***-congruence on A? Let

$$X = \{x_i \mid i \in I\} \qquad Y = \{y_i \mid i \in I\}$$

be a pair of similarly indexed subsets of A (that is, over the same index set). We write

$$X \sim Y$$

if

$$(\forall i \in I)[x_i \sim y_i]$$

that is if X and Y are point-wise equivalent.

3.3 DEFINITION. An equivalence relation $\sim$ on a $\bigvee$-semilattice A is a ***Sup***-congruence if we have

$$X \sim Y \Longrightarrow \bigvee X \sim \bigvee Y$$

for each similarly indexed pair X, Y of subsets of A. ■

Let

$$A \xrightarrow{\quad f \quad} B$$

be a $\bigvee$-morphism. As in the ***Set***-case, the kernel of f is the equivalence relation $\approx$ on A given by

$$x \approx y \Longleftrightarrow f(x) = f(y)$$

for $x, y \in A$. The morphism property translates into the following.

3.4 LEMMA. *The kernel of a $\bigvee$-morphism is a **Sup**-congruence on the source algebra.*

Our job here is to produce a converse of this result, We show that every ***Sup***-congruence is obtained from a $\bigvee$-morphism.

Consider how we might do that.

Let A be a $\bigvee$-semilattice and let $\sim$ be a ***Sup***-congruence on A. As in the ***Set***-case, consider the function

$$A \xrightarrow{\quad \eta \quad} A/{\sim}$$

which sends each element to its block. The idea is to furnish $A/{\sim}$ as a $\bigvee$-semilattice in such a way that η becomes a $\bigvee$-morphism. How might we furnish $A/{\sim}$ with a supremum operation? Consider a subset $\mathcal{X}$ of $A/{\sim}$. By choosing block representatives we can view this as

$$\mathcal{X} = \eta^{\rightarrow}[X]$$

for some subset X of A. The idea is to define

$$\bigvee \mathcal{X} = \eta(\bigvee X)$$

which automatically ensures that η is a $\bigvee$-morphism. There is, of course, some work to be done. We must show that the definition of this 'supremum' operation in $A/{\sim}$ is independent of the choice of block representatives. We also have to show that it is a supremum operation, which mean we should first set up a partial ordering on $A/{\sim}$ and show that η is monotone.

All this can be done, but it's messy and, thankfully, avoidable. The cause of the mess is the use of arbitrary block representatives. This can be cleaned up using special block representatives.

3.5 LEMMA. *Let $\sim$ be a **Sup**-congruence on a $\bigvee$-semilattice A. Then each $\sim$-block has a unique largest member.*

Proof. Consider any $a \in A$. Let

$$X = \{x_i \mid i \in I\}$$

be an indexing of the block to which a belongs. Also let

$$Y = \{y_i \mid i \in I\}$$

where $y_i = a$ for each $i \in I$. By construction we have

$$X \sim Y$$

so that (since $\sim$ is a ***Sup***-congruence)

$$\bigvee X \sim \bigvee Y = a$$

to show that $\bigvee X$ is the largest member of the block in question. ■

In the standard congruence situation each block is handled by some block representative, some member of that block. In general any one representative is no better than any other. However, Lemma 3.5 shows that each block of a ***Sup***-congruence has a special representative, and that is the one we use. We also use the operator which attaches to each element its largest mate.

3.6 DEFINITION. Let $\sim$ be a ***Sup***-congruence on a $\bigvee$-semilattice A. The selector for $\sim$ is the operator $j : A \longrightarrow A$ given by

$$j(a) = \bigvee\{x \in A \mid a \sim x\}$$

for each $a \in A$. ■

By construction, if j is the selector of the congruence $\sim$ on A then we have

$$(\bullet) \quad a \sim j(a) \qquad\qquad (\bullet\bullet) \quad x \sim a \Longrightarrow x \le j(a)$$

for all $a, x \in A$. These two properties characterize being the selector of $\sim$, and enable us to see selectors in a different light. Recall that a closure operation on A is a function $j : A \longrightarrow A$ which is inflationary, monotone, and idempotent, that is

$$(i) \quad a \le j(a) \qquad (m) \quad b \le a \Longrightarrow j(b) \le j(a) \qquad (c) \quad j(j(a)) = j(a)$$

for all $a, b \in A$.

3.7 LEMMA. *Let A be a $\bigvee$-semilattice.*
The selectors on A are precisely the closure operators.
Each closure operation is the selector of precisely one congruence.
There is a bijective correspondence between ***Sup****-congruence relations and closure operators on A.*

Proof. Suppose that j is the selector of the congruence $\sim$ on A. We use the two properties $(\bullet)$ and $(\bullet\bullet)$ from above to show that j is a closure operation on A.

For each $a \in A$ we have $a \sim a$, so that $(\bullet\bullet)$ gives $a \le j(a)$, to show that j is inflationary.

Consider $a, b \in A$ with $b \le a$. We have

$$a \sim j(a) \qquad b \sim j(b)$$

by $(\bullet)$, so that

$$j(a) \vee j(b) \sim a \vee b = a$$

by the congruence property, and hence

$$j(b) \le j(a) \vee j(b) \le j(a)$$

by $(\bullet\bullet)$. This shows that j is monotone.

Consider any $a \in A$ and let $b = j(j(a))$. We have

$$a \sim j(a) \sim b$$

by two uses of $(\bullet)$, so that $b \sim a$, and hence $b \le j(a)$ by $(\bullet\bullet)$. The converse comparison $j(a) \le b$ holds since j is inflationary. This shows that j is idempotent.

These three small arguments show that the selector j is a closure operation.

Suppose j is a closure operation on A. We show that j is the selector of at least one congruence on A. To this end consider the relation $\sim$ on A given by

$$x \sim y \Longleftrightarrow j(x) = j(y)$$

for $x, y \in A$. Trivially, this is an equivalence relation on A. To show it is a congruence relation suppose $X \sim Y$ for two similarly indexed subsets X, Y of A. For each $y \in Y$ there is some $x \in X$ with $x \sim y$, so that

$$y \le j(y) = j(x) \le j(\bigvee X)$$

by the inflationary and monotone properties of j. Thus

$$\bigvee Y \leq j(\bigvee X)$$

to give

$$j(\bigvee Y) \leq j(j(\bigvee X)) \leq j(\bigvee X)$$

by the monotone and idempotent properties of j. By symmetry this gives

$$j(\bigvee X) = j(\bigvee Y) \quad \text{and hence} \quad \bigvee X \sim \bigvee Y$$

as required to show that $\sim$ is a congruence on A.

The idempotent and inflationary properties of j ensure that ($\bullet$) and ($\bullet\bullet$) hold, and hence j is the selector of $\sim$.

Trivially (by definition) each congruence has just one selector. Thus, to complete the whole proof it suffices to show that each closure operation is the selector of just one congruence. We show that if j is the selector of the congruence $\sim$ then

$$x \sim y \iff j(x) = j(y)$$

for each $x, y \in A$.

Consider $x, y \in A$ with $x \sim y$. By hypothesis, j is the selector of $\sim$, and hence

$$x \leq j(y) \qquad y \leq j(x)$$

by two uses of ($\bullet\bullet$). Thus $j(x) = j(y)$ by the monotone and idempotent properties of j.

Consider $x, y \in A$ with $j(x) \sim j(y)$. By hypothesis, j is the selector of $\sim$, and hence

$$x \sim j(x) = j(y) \sim y$$

by two uses of ($\bullet$). Thus $x \sim y$ since $\sim$ is an equivalence relation. ■

This result enables us to do away with any use of block representatives, and to replace the use of congruences by the use of closure operators. In time this will give us a collection of powerful techniques for analysing $\bigvee$-semilattices, but for now we concentrate on producing a ***Sup***-version of Theorem 3.1.

3.8 DEFINITION. Let A be a $\bigvee$-semilattice.

A subset $F \subseteq A$ is $\bigwedge$-closed if $\bigwedge X \in F$ for each $X \subseteq F$.

For a closure operation j on A we set

$$A_j = j^{\rightarrow}(A) = \{x \in A \mid j(x) = x\}$$

to obtain the set of fixed elements of j. ■

Observe that if $F \subseteq A$ is $\bigwedge$-closed then, by considering $\emptyset \subseteq F$, we have $\top = \bigwedge \emptyset \in F$, and hence F is non-empty. These $\bigwedge$-closed sets have a more important property.

3.9 LEMMA. *Let A be a $\bigvee$-semilattice.*

For each closure operation j on A the subset A_j is $\bigwedge$-closed.

For each $\bigwedge$-closed subset $F \subseteq A$ there is a unique closure operation j on A with $F = A_j$

There is a bijective correspondence between closure operations on A and $\bigwedge$-closed subsets.

Proof. Let j be a closure operation on A and consider $X \subseteq A_j$. For each $x \in X$ we have $\bigwedge X \leq x$ so that

$$j(\bigwedge X) \leq j(x) = x$$

and hence $j(\bigwedge X) \leq \bigwedge X$ to show that $\bigwedge X \in A_j$.

Let F be $\bigwedge$-closed in A. For each $a \in A$ set

$$j(a) = \bigwedge\{x \in F \mid a \leq x\}$$

to obtain an operator j on A. Almost trivially, this j is inflationary and monotone. Furthermore, for each $a \in A$ we have

$$j(a) \in F \qquad a \in F \Longrightarrow j(a) = a$$

(where the $\bigwedge$-closed property gives the left hand condition). In combination these two conditions give

$$j(j(a)) = j(a)$$

(for each $a \in A$), to show that j is idempotent, and hence is a closure operation.

The right hand condition gives $F \subseteq A_j$. Conversely, if $a \in A_j$ then $a = j(a) \in F$ by the left hand condition, to show $F = A_j$.

Finally, suppose we have $A_j = A_k$ for closure operations on A. Consider any $a \in A$. We have $j(a) \in A_j = A_k$, so that

$$k(j(a)) = j(a)$$

and hence

$$k(a) \leq k(j(a)) = j(a)$$

by the inflationary property of j and the monotone property of k. A similar argument gives $j(a) \leq k(a)$, and hence $j = k$. ■

We now have enough background to begin the ***Sup***-analogue of Theorem 3.1.

Let A be a $\bigvee$-semilattice, and let j be a closure operation on A. Remember that we think of j as a more socially acceptable version of a ***Sup***-congruence on A. Consider the function

$$\begin{array}{rcl} A & \xrightarrow{\ j^*\ } & A_j \\ a & \longmapsto & j(a) \end{array}$$

(and do not confuse this with the very similar function j). Since A_j is a subset of A it inherits a comparison from A, so A_j is at least a poset. We show a bit more.

3.10 LEMMA. *Consider the situation above.*

For each subset $X \subseteq A_j$, the element $j(\bigvee X)$ is the supremum of X in A_j.

The poset $(A_j, \leq)$ is complete.

*The assignment j^* is a **Sup**-morphism.*

Proof. Consider $X \subseteq A_j$. For each $x \in X$ we have

$$x \le \bigvee X \le j(\bigvee X) \in A_j$$

so that $j(\bigvee X)$ is an upper bound of X in A_j. Let $a \in A_j$ be any upper bound of X in A_j. We have

$$\bigvee X \le a$$

(in A) so that

$$j(\bigvee X) \le j(a) = a$$

to show that $j(\bigvee X)$ is the least upper bound of X in A_j.

This also shows that A_j is complete (as a poset).

To show that j^* is a ***Sup***-morphism we require

$$j^*(\bigvee X) = j(\bigvee j^{*\rightarrow}(X))$$

for each $X \subseteq A$. The element on the right hand side is the supremum of the subset $j^{*\rightarrow}(X))$ in A_j. This required equality rephrases as

$$j(\bigvee X) = j(\bigvee \{j(x) \mid x \in X\})$$

and we verify this via two comparisons.

For each $x \in X$ we have

$$x \le j(x) \le \bigvee \{j(x) \mid x \in X\})$$

so that

$$\bigvee X \le \bigvee \{j(x) \mid x \in X\})$$

and hence

$$j(\bigvee X) \le j(\bigvee \{j(x) \mid x \in X\})$$

since j is monotone.

Conversely, for each $x \in X$ we have

$$x \le \bigvee X$$

so that

$$j(x) \le j(\bigvee X)$$

to give

$$\bigvee \{j(x) \mid x \in X\} \le j(\bigvee X)$$

and hence

$$j(\bigvee \{j(x) \mid x \in X\}) \le j^2(\bigvee X) = j(\bigvee X)$$

since j is monotone and idempotent. ■

By Lemma 3.4 each ***Sup***-morphism

$$A \xrightarrow{\ f\ } B$$

has a kernel $\approx$ given by

$$x \approx y \iff f(x) = f(y)$$

for $x, y \in A$. This, of course, views the kernel as a congruence. By Lemma 3.7 we may re-view this congruence as a closure operation k given by

$$k(x) = k(y) \iff x \approx y$$

for $x, y \in A$. These two characterizations enable us to move directly from the morphism f to the closure operation k.

3.11 DEFINITION. For each $\boldsymbol{Sup}$-morphism f, as above, the kernel of f is the unique closure operation k on A (the source of f) such that

$$k(x) = k(y) \iff f(x) = f(y)$$

for all $x, y \in A$. ■

This definition uniquely specifies the kernel of a morphism, but doesn't really tell us what it is. To discover that we remember that each $\boldsymbol{Sup}$-morphism has a right adjoint

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B$$

which, as I promised you earlier, has its uses.

3.12 LEMMA. *For each* $\mathbf{Sup}$*-morphism* $f^* \dashv f_*$*, as above, the kernel is the composite* $f_* \circ f^*$.

Proof. By the general properties of poset adjunctions we know that $k = f_* \circ f^*$ is a closure operation on the source A. Thus it suffices to show that

$$(f_* \circ f^*)(x) = (f_* \circ f^*)(y) \iff f^*(x) = f^*(y)$$

holds for all $x.y \in A$. The implication $\Leftarrow$ is immediate, and the converse holds since $f^* \circ f_* \circ f_* = f_*$. ■

With a little bit more work we can characterize the kernel directly in terms of the morphism without using the right adjoint.

3.13 COROLLARY. *For each* $\mathbf{Sup}$*-morphism* f*, as above, the kernel is the closure operation* k *such that*

$$x \leq k(a) \iff f(x) \leq f(a)$$

for each $x, a \in A$.

Proof. We have

$$x \leq f_*(y) \iff f^*x \leq y$$

for each $x \in A$ and $y \in B$. Setting $y = f(a) = f^*(a)$ gives the required result. ■

Finally we can prove the $\bigvee$-refinement of Theorem 3.1.

3.14 THEOREM. *Let j be a closure operation on the $\bigvee$-semilattice A, and let*

$$A \xrightarrow{\ f\ } B$$

*be a **Sup**-morphism with kernel k. Suppose $j \leq k$. Then there is a unique **Sup**-morphism $f^\sharp$ such that*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j^*"'] & & B \\ & A_j \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

Proof. Since j^* is surjective, there can be at most one such fill-in morphism $f^\sharp$.
For each $a \in A$ we have

$$k(k(a)) = k(a)$$

so that

$$f(k(a)) = f(a)$$

by one of the characteristic properties of the kernel k of f. We also have

$$a \leq j(a) \leq k(a)$$

so that

$$f(a) \leq f(j(a)) \leq f(k(a)) = f(a)$$

and hence

$$f(j(a)) = f(a)$$

by the previous observation.
For each $x \in A_j$ set

$$f^\sharp(x) = f(x)$$

to obtain a function $f^\sharp : A_j \longrightarrow B$. We have just seen that

$$(f^\sharp \circ j^*)(a) = f(j(a)) = f(a)$$

for each $a \in A$, so it suffices to show that $f^\sharp$ is a ***Sup***-morphism.
We require

$$f^\sharp(\breve{\bigvee} X) = \bigvee f^{\sharp\rightarrow}(X)$$

for each $X \subseteq A_j$. Here $\breve{\bigvee}$ is the supremum operation on A_j. This condition unravels as

$$f(j(\bigvee X)) = f^{\rightarrow}(X)$$

which, since $f \circ j = f$, reduces to the given morphism property of f. ■

As you can probably guess, we are going to refine this result even further by replacing ***Sup*** by ***Frm***.

3.3 Quotients in *Frm*

The results of Subsection 3.2 are important in a wider context, but here we are primarily concerned with frames. Each frame morphism

$$A \xrightarrow{f} B$$

is a $\bigvee$-morphism of a special kind. As such it has a kernel j given by

$$x \leq j(a) \Longleftrightarrow f(x) \leq f(a)$$

for $x, a \in A$. This j is a closure operation on A, of a special kind. Our job in this subsection is to isolate and begin to investigate these special closure operations

3.15 DEFINITION. A nucleus on a frame A is a closure operation j on A such that

$$j(a \wedge b) = j(a) \wedge j(b)$$

for all $a, b \in A$. ■

If you think about it you have already seen one family of examples of nuclei. We will return to those examples later.

3.16 LEMMA. *The kernel of a frame morphism is a nucleus on the source.*

Proof. Consider a frame morphism f with its kernel j, as above. For $a, b \in A$ (the source of f and carrier of j) we have

$$f(a \wedge b) = f(a) \wedge f(b)$$

and we require a corresponding equality for j. For each $x \in A$ the characterizing property of j gives

$$\begin{aligned} x \leq j(a \wedge b) &\Longleftrightarrow f(x) \leq f(a \wedge b) = f(a) \wedge f(b) \\ &\Longleftrightarrow f(x) \leq f(a) \text{ and } f(x) \leq f(b) \\ &\Longleftrightarrow x \leq j(a) \text{ and } x \leq j(b) \qquad \Longleftrightarrow x \leq j(a) \wedge j(b) \end{aligned}$$

so that

$$j(a \wedge b) = j(a) \wedge j(b)$$

as required. ■

Of course, we want to show that every nucleus arises as the kernel of a frame morphism. To do that we refine the ideas of Defintion 3.8.

3.17 DEFINITION. A fixed set of a frame A is a $\bigwedge$-closed subset $F \subseteq A$ such that

$$a \in A_j \Longrightarrow (x \supset a) \in A_j$$

for all $a, x \in A$. ■

By Lemma 3.9 the closure operations on a frame A correspond to the $\bigwedge$-subsets. This correspondence refines as follows.

3.18 LEMMA. *A closure operation j on a frame A is a nucleus precisely when its set A_j of fixed elements is a fixed set.*

Proof. Suppose first that j is a nucleus and consider

$$y = (x \supset a)$$

for $a \in A_j$ and arbitrary $x \in A$. We have $x \wedge y \leq a$, so that

$$x \wedge j(y) \leq j(x) \wedge j(y) \leq j(x \wedge y) \leq j(a) = a$$

to give

$$j(y) \leq (x \supset a) = y$$

and hence $y \in A_j$. This shows that A_j is a fixed set.

Conversely, suppose A_j is a fixed set, and consider arbitrary $x, y \in A$. It suffices to show that

$$j(x) \wedge j(y) \leq j(x \wedge y)$$

(since the converse comparison is a consequence of the monotonicity of j). Let $a = j(x \wedge y)$, so that $a \in A_j$. We have

$$x \wedge y \leq j(x \wedge y) = a$$

so that

$$y \leq (x \supset a) \in A_j$$

to give

$$j(y) \leq (x \supset a)$$

and hence

$$x \wedge j(y) \leq a$$

holds. A repeat of this argument (with x and y playing different roles) gives

$$j(x) \wedge j(y) \leq a$$

which is the required result. ■

Each nucleus j on a frame A has a fixed set A_j and, by Lemma 3.10, j induces a $\bigvee$-morphism from A ro A_j with j as its kernel. This refines as follows.

3.19 LEMMA. *Let j be a nucleus on the frame A. The fixed set A_j is a frame, and the assignment*

$$\begin{array}{ccc} A & \xrightarrow{j^*} & A_j \\ a & \longmapsto & j(a) \end{array}$$

is a frame morphism.

Proof. By Lemma 3.10 the poset $(A_j, \leq)$ is complete. Thus, by Lemma 1.7, it suffices to show that A_j carries an implication. But Lemma 3.18 shows that A_j is closed under the implication carried by A, and it is easy to check that this provides an implication on A_j.

By Lemma 3.10 the assignment j^* is a ***Sup***-morphism, thus we require

$$j^*(x \wedge y) = j^*(x) \wedge j^*(y)$$

for $x, y \in A$. This is immediate since j is a nucleus. ■

With this we can prove the refinement of Theorem 3.14.

3.20 THEOREM. *Let j be a nucleusn on the frame A, and let*

$$A \xrightarrow{f} B$$

be a frame morphism with kernel k. Suppose $j \leq k$. Then there is a unique frame morphism $f^\sharp$ such that

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j^*"'] & & B \\ & A_j \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

Proof. By Theorem 3.14 there is a unique ***Sup***-morphism $f^\sharp$ for which the triangle commutes. Thus it suffices to show that this $f^\sharp$ satisfies

$$f^\sharp(x \wedge y) = f^\sharp(x) \wedge f^\sharp(y)$$

for $x, y \in A_j$. This is an immediate consequence of the definition of $f^\sharp$ since $A_j \subseteq A$ and f is a frame morphism. ■

The results of this section show that we ought to find out a lot more about nuclei. We begin that in the next section.

4 Nuclei on frames

As in Definition 3.15, a nucleus j on a frame A is a closure operation that passes across binary meets. The results of Subsection 3.3 suggest that these nuclei have a role to play in the analysis of frames. In fact, as we will see later, they have a considerable impact on the whole subject. In this section we look at a few of the basic examples and results.

4.1 DEFINITION. For each element a of a frame A we set

$$\mathsf{u}_a(x) = (a \vee x) \qquad \mathsf{v}_a(x) = (a \supset x) \qquad \mathsf{w}_a(x) = ((x \supset a) \supset a)$$

for each $x \in A$, to obtain three operators on A. ■

The operators w_a are the same as those introduced in Definition 1.10, and we now see that Lemma 1.11 shows that each of these is a nucleus. That is the harder part of the proof of the following.

4.2 LEMMA. *For each frame A and $a \in A$, the three operators $\mathsf{u}_a, \mathsf{v}_a, \mathsf{w}_a$ are nuclei on A.*

Several straight forward calculations show that u_a and v_a are nuclei, but let's look at another proof.

The kernel of each frame morphism is a nucleus on the source. This can be a useful way of showing that an operator is a nucleus. Here is a simple example of this technique.

4.3 EXAMPLE. Let A be a frame and let $a \in A$ be an arbitrary element. Consider the two principal sections

$$[a, \top] \qquad [\bot, a]$$

above and below a. Each of these is a complete poset in its own right. In fact, after a few moment's thought we see that each is a frame in its own right (but neither is a subframe of A, unless a takes an extreme position). However, each of the two assignments

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & [a, \top] \\ y & \longmapsto & a \vee y \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\ f\ } & [\bot, a] \\ y & \longmapsto & a \wedge y \end{array}$$

is a surjective frame morphism. (You should check this and notice how the FDL is used.) Each of these morphisms has a kernel k given by

$$y \le k(x) \Longleftrightarrow a \vee y \le a \vee x \qquad\qquad y \le k(x) \Longleftrightarrow a \wedge y \le a \wedge x$$

respectively. These give

$$k = \mathsf{u}_a \qquad\qquad k = \mathsf{v}_a$$

to show that u_a and v_a are nuclei on A.

The fixed set A_{u_a} of u_a is precisely the interval $[a, \top]$ we started from. This fixed set A_{v_a} of v_a is *not* the interval $[\bot, a]$, but is is canonically isomorphic to this interval. ■

In a spatial situation the $\mathsf{u}_\bullet$ and $\mathsf{v}_\bullet$ nuclei have a common generalization.

4.4 DEFINITION. Let S be a space with topology $\mathcal{O}S$, and let $E \subseteq S$. We set

$$[E](U) = (E \cup U)^\circ$$

for each $U \in \mathcal{O}S$, to produce an operator on $\mathcal{O}S$. ■

Observe that for an open set $A \in \mathcal{O}S$ we have

$$[A](U) = A \cup U$$

for each $U \in \mathcal{O}S$, and so $[A] = \mathsf{u}_A$ on $\mathcal{O}S$. Similarly, we have

$$[A'](U) = (A' \cup U)^\circ = (A \supset U)$$

for each $U \in \mathcal{O}S$, and so $[A'] = \mathsf{v}_A$ on $\mathcal{O}S$. Later we will see that u_a and v_a are complementary on an arbitrary frame.

A proof of the following is straight forward.

4.5 LEMMA. *For each space S with topology $\mathcal{O}S$ and each $E \subseteq S$, the operator $[E]$ is a nucleus on $\mathcal{O}S$.*

Proof. Trivially, the operator $[E]$ is inflationary and monotone. For $F, G \subseteq S$ we have

$$(F \cap G)^\circ = F^\circ \cap G^\circ$$

and hence $[E]$ is a pre-nucleus. Finally, for $U \in \mathcal{O}S$ we have

$$[E]^2(U) = [E]([E](U)) = (E \cup (E \cup U)^\circ)^\circ \subseteq (E \cup E \cup U)^\circ = (E \cup U)^\circ = [E](U)$$

to show that $[E]$ is idempotent. ■

As with the $\mathsf{u}_\bullet$ and $\mathsf{v}_\bullet$ nuclei, it is instructive to see $[E]$ exhibited as the kernel of a frame morphism. To do this consider a continuous map

$$T \xrightarrow{\ \phi\ } S$$

from a space T to a space S. From Lemma 1.17 this induces a frame morphism and its adjoint

$$\mathcal{O}S \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{O}T$$

where

$$\phi^*(U) = \phi^{\leftarrow}(U) \qquad \phi_*(W) = \phi^{\rightarrow}(W')^{-\prime}$$

for each $U \in \mathcal{O}S$ and $V \in \mathcal{O}T$. This morphism has a kernel

$$\phi_* \circ \phi^*$$

which has a simpler description.

4.6 LEMMA. *For the situation above the kernel of $\phi^{\leftarrow}$ is $[E]$ where $E = T - \phi^{\rightarrow}(S)$, the complement of the range of ϕ.*

Proof. Let k be the kernel of $\phi^{\leftarrow}$. Remembering the characterization k given by Corollary 3.13, for each $U, V \in \mathcal{O}S$ we have

$$\begin{aligned} V \subseteq k(U) &\iff \phi^{\leftarrow}(V) \subseteq \phi^{\leftarrow}(U) \\ &\iff (\forall t \in T)[\phi(t) \in V \implies \phi(t) \in U] \\ &\iff (\forall s \in \phi^{\rightarrow}(T))[s \in V \implies s \in U] \\ &\iff \phi^{\rightarrow}(T) \cap V \subseteq U \\ &\iff V \subseteq E \cup U \qquad\qquad \iff V \subseteq [E](U) \end{aligned}$$

to give the required result. ■

This idea deserve a bit of terminology.

4.7 DEFINITION. A nucleus on a topology $\mathcal{O}S$ is spatially induced if it has the form $[E]$ for some $E \subseteq S$. ■

In general for a space there are nuclei on $\mathcal{O}S$ that are not spatially induced. In fact, as we will see in [6] every nucleus on a topology is spatially induced precisely when the parent space has a certain amount of pathology. Thus we could say that it is those nuclei on a topology that are not spatially induced that are the interesting ones.

We first met the nuclei $\mathsf{w}_\bullet$ in Subsection 1.4, where I said they are important gadgets. Here is one (but not the only) reason for saying that.

4.8 LEMMA. *For each frame A and element $a \in A$, when viewed as a frame the fixed set*

$$A_{\mathsf{w}_a}$$

is boolean.

Proof. Observe that

$$a = \mathsf{w}_a(\bot)$$

is the bottom of A_{w_a}. Consider any $x \in A_{\mathsf{w}_a}$. We must produce some $y \in A_{\mathsf{w}_a}$ with

$$x \wedge y = a \qquad \mathsf{w}_a(x \vee y) = \top$$

where, of course, $\mathsf{w}_a(x \vee y) = \top$ is the join of x and y in A_{w_a}.

Let

$$y = (x \supset a)$$

so that

$$\mathsf{w}_a(y) = \mathsf{w}_a(x \supset a) = (\mathsf{w}_a(x) \supset a) = (x \supset a) = y$$

By Lemma 1.11. In particular, $y \in A_{\mathsf{w}_a}$. Also

$$x \wedge y = x \wedge a = a$$

since $a \leq x$. Thus it remains to deal with the join in A_{w_a}.

Using Lemma 1.9(iv) we have

$$((x \vee y) \supset a) = (x \supset a) \wedge (y \supset a) = y \wedge (y \supset a) = a$$

since $a \leq y$. Thus

$$\mathsf{w}_a(x \vee y) = (a \supset a) = \top$$

for the required result. ■

Each nucleus w_a gives a boolean quotient. What is more interesting is that ever boolean quotient arises in this way.

4.9 THEOREM. *Suppose*

$$A \xrightarrow{f} B$$

is a surjective frame morphism with a boolean target B. Then the kernel k of f is w_a where $a = k(\bot)$.

Proof. In this proof it is convenient to write

$$\bot_A \quad \top_A \qquad \bot_B \quad \top_B$$

for the extremes of the two frames. Thus $a = k(\bot_A)$, and satisfies $f(a) = \bot_B$.

We first check that

$$(x\!\uparrow) \quad f(x) \vee f(x \supset a) = \top_B$$
$$(x\!\downarrow) \quad f(x) \wedge f(x \supset a) = \bot_B$$

for each $x \in A$.

For $(x\!\downarrow)$ we have

$$f(x) \wedge f(x \supset a) = f(x \wedge (x \supset a)) = f(x \wedge a) \leq f(a) = \bot_B$$

to give the equality. This does not use any properties of B.

For $(x\!\uparrow)$ we know that $f(x)$ has a complement in B and this is the image of some element of A. Thus we have

$$f(x) \vee f(z) = \top_B$$
$$f(x) \wedge f(z) = \bot_B$$

for some $z \in A$. The lower one of these gives

$$f(x \wedge z) \leq f(\bot_A)$$

so that

$$x \wedge z \leq k(\bot_A) = a$$

(since k is the kernel of f) and hence

$$z \leq (x \supset a)$$

which with the upper one leads to

$$f(x) \vee f(x \supset a) \geq f(x) \vee f(z) = \top_B$$

as required.

From $(x\!\uparrow, x\!\downarrow)$ we have

$$b \leq f(x) \iff b \wedge f(x \supset a) = \bot_B$$

for each $b \in B$. Thus, for each $y \in A$ we have

$$\begin{aligned} y \leq k(x) &\iff f(y) \leq f(x) \\ &\iff f(y) \wedge f(x \supset a) = \bot_B = f(\bot_A) \\ &\iff f(y \wedge (x \supset a)) \leq f(\bot_A) \\ &\iff y \wedge (x \supset a) \leq k(\bot_A) = a \\ &\iff y \leq (x \supset a) \supset a \qquad\qquad \iff y \leq \mathsf{w}_a(x) \end{aligned}$$

to give the required result. ■

Each nucleus j on the frame A produces a quotient

$$A \xrightarrow{\ j^*\ } A_j$$

to the frame A_j of element fixed by j. Furthermore, each quotient of A arises in this way (up to a unique isomorphism over A). We have seen that u_a and v_a arise from

$$A \longrightarrow [a, \top] \qquad\qquad A \longrightarrow [\bot, a]$$

respectively, where a little bit of care is needed with the right hand quotient. What about

$$A \xrightarrow{j^*} A_{\mathsf{w}_a}$$

for arbitrary a? To answer that we first look at the particular case $a = \bot$.

For each $x \in A$ we have

$$\mathsf{w}_\bot(x) = ((x \supset \bot) \supset \bot)\neg\neg x$$

so that $\mathsf{w}_\bot$ is just double negation on A. In particular

$$A_{\mathsf{w}_\bot} = A_{\neg\neg} = \{x \in A \mid \neg\neg x = x\}$$

is the set of regular elements of A. This is converted into a frame using

$$x \check{\vee} y = \neg\neg(x \vee y) = \neg(\neg x \wedge \neg y)$$

as the join of x and y in $A_{\neg\neg}$. This is well known in topological circles.

4.10 EXAMPLE. For the topology $\mathcal{O}S$ on a space S the frame $(\mathcal{O}S)_{\neg\neg}$ is the complete boolean algebra of regular open sets of S.

We know that a regular element of $\mathcal{O}S$ is just a regular open set. For two such sets U, V the join in $(\mathcal{O}S)_{\neg\neg}$ is given by

$$U \vee V = \neg\neg(U \cup V) = (U \cup V)^{-\circ}$$

which is precisely the way we convert the regular open sets into a boolean algebra. ■

For each $a \in A$ the quotient A_{w_a} is a complete boolean algebra. We have just seen that

$$A_{\neg\neg} = A_{\mathsf{w}_{bot}}$$

is the analogue of the boolean algebra of regular pen sets. To deal with A_{w_a} for arbitrary a we look at the interval

$$[a, \top]$$

as a frame. What is the negation of an element $x \in [a, \top]$ in $[a, \top]$? It is that element $y \in [a, \top]$ such that

$$z \le y \iff z \wedge x \le a$$

for $z \in [a, \top]$. In other words it is $(x \supset a)$, which does live in $[a, \top]$. Furthermore, the double negation of x in $[a, \top]$ is just $\mathsf{w}_a(x)$. Thus

$$[a, \top]_{\neg\neg} = A_{\mathsf{w}_a}$$

to verify the following.

4.11 LEMMA. *For each element $a \in A$ of the frame A the kernel of the obvious composite morphism*

$$A \longrightarrow [a, \top] \longrightarrow [a, \top]_{\neg\neg}$$

is w_a.

We will return to the nuclei w_a many times during the course of these sets of notes. We conclude this section with a little result which is occasionally useful.

4.12 LEMMA. *Let j be a nucleus on a boolean frame A. Then $j = \mathsf{u}_a$ where $a = j(\bot)$.*

Proof. For each $x \in A$ we have

$$a = j(\bot) \leq j(x) \qquad x \leq j(x)$$

so that

$$a \vee x \leq j(x)$$

and it suffices to show the converse comparison.

Consider the negation $\neg x$ of x in A. We have

$$x \wedge \neg x = \bot$$

so that

$$j(x) \wedge \neg x \leq j(x) \wedge j(\neg x) = j(x \wedge \neg x) = j(\bot) = a$$

and hence

$$j(x) \leq a \vee x$$

since A is boolean. ∎

In a way, this shows that, as frames, complete boolean algebras don't hold much interest.

5 Various reflections

In the course of these notes we hyave met several functors, some explicitly and some implicitly. Let's collect together in one place the important ones. As we do this you may complain that some of the categories involved have not yet been defined. Don't wory. You will have a rough idea of what each category is, and precise definition are given in the following subsections at the appropriate place.

By definition, each frame is both a $\wedge$-semilattice and a $\bigvee$-semilattice both carried by the same poset. Thus, as indicated in Subsection 1.1, there are two functors

$$\boldsymbol{Meet} \longleftarrow \boldsymbol{Frm} \qquad \boldsymbol{Sup} \longleftarrow \boldsymbol{Frm}$$

where each forgets part of the carried structure. (You will soon see why these have been written from right to left.) The functor to ***Meet*** can be factorized through the category ***Dlt*** of d-lattices

$$\boldsymbol{Meet} \longleftarrow \boldsymbol{Dlt} \longleftarrow \boldsymbol{Frm}$$

where each functor forgets both stucture and property in various degrees. There is also an extremely forgetful functor

$$\boldsymbol{Set} \longleftarrow \boldsymbol{Frm}$$

to the category of sets and functors. In this section we consider the problem of providing reflections for most of these functors.

Before we begin that there are two other functors

$$\boldsymbol{Frm} \longrightarrow \boldsymbol{Top} \qquad\qquad \boldsymbol{CBA} \longrightarrow \boldsymbol{Frm}$$

that are worth mentioning. These are significantly different in several ways.

The first, from ***Frm*** to ***Top*** appears in Subsection 1.2. This is *contravriant*, so it does not make sense to think of it as forgetful. Nevertheless, it does have an adjoint, that is it is part of a contravariant adjunction between ***Frm*** and ***Top***. This the central topic of [5].

The second, from ***CBA*** to ***Frm*** appears in Subsection 1.3, and is forgetful. However, it does *not* have a reflection (for set theoretical reasons). This is the central topic of [7].

5.1 Reflections in general

It seems obvious that we should start with a definition of 'forgetful' functor. There is such a definition but it isn't much use. Here we use the terminolgy in an informal way. We don't need a precise definition since we only consider a finite number of examples, and we will all agree that these functors are forgetful.

For us a functor

$$\boldsymbol{Gauch} \xleftarrow[\mathcal{G}]{} \boldsymbol{Fine}$$

is forgetful if '$\mathcal{G}$' can be omitted in calculations without causing too much confusion. Thus $\mathcal{G}$ converts each ***Fine***-object A into a ***Gauch***-object $\mathcal{G}A$ which we can think of as A viewed as a ***Gauch***-object. For instance, each frame can be viewed as a $\bigvee$-semilattice, a d-lattice, a $\wedge$-semilattice, or even as a set. We simply ignore some of its structure or property. Similarly $\mathcal{G}$ converts each ***Fine***-arrow f into a ***Gauch***-arrow $\mathcal{G}(f)$ which we can think of as f viewed as a ***Gauch***-arrow. For instance, each frame norphism f can be viewed as a $\bigvee$-morphism, a lattice morphism, a $\wedge$-semilattice morphism, or even just as a function. We simply ignore some of its preservation properties.

Suppose we have a functor $\mathcal{G}$, as above, which we can think of forgetful. A reflection (over $\mathcal{G}$) or a left adjoint $\mathcal{F} \dashv \mathcal{G}$ is a functor

$$\boldsymbol{Gauch} \xrightarrow{\mathcal{F}} \boldsymbol{Fine}$$

which iteracts with $\mathcal{G}$ in a certain way. For what we do here the following is the most appropriate characterization. Remember that a forgetful functor need not be mentioned in calculations.

5.1 DEFINITION. Let $\mathcal{G}$, as above, be a forgetful functor. Let S be a ***Gauch***-object.

A reflection of S into ***Fine*** is a selected ***Gauch***-arrow

$$S \xrightarrow{\eta_S} \mathcal{F}S$$

to a selected ***Fine***-object $\mathcal{F}S$ with the following universal property.

For each ***Gauch***-arrow

$$S \xrightarrow{f} A$$

to a ***Fine***-object A there is a unique ***Fine***-arrow

$$\mathcal{F}S \xrightarrow{f^\sharp} A$$

such that

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\eta_S"'] & & A \\ & \mathcal{F}S \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes.

We say ***Fine*** is reflective in ***Gauch*** (via $\mathcal{G}$) if each ***Gauch***-object has a reflection in ***Fine***. ■

Notice the stucture of this definition. We first consider when a single ***Gauch***-object S can be reflected into ***Fine***, and the we impose that conditions on all ***Gauch***-objects. There are some forgetful functors for which only certain objects can be reflected. An analysis of one such example is the central topic of [7]. Here we consider only cases where all objects can be reflected.

Notice that in Definition 5.1 I omitted to mention the functor $\mathcal{G}$ in certain places. For instance, the selected arrow shold be

$$S \xrightarrow{\eta_S} (\mathcal{G} \circ \mathcal{F})S$$

since it lives in ***Gauch***. If you find this confusing, then write out the definition in full and try to prove the following standrad result concerning adjunctions.

5.2 THEOREM. *Let $\mathcal{G}$ be a forgetful functor, as above. Suppose each **Gauch**-objet has a reflection*

$$S \xrightarrow{\eta_S} (\mathcal{G} \circ \mathcal{F})S$$

*into **Fine** Then*

$$S \longmapsto \mathcal{F}S$$

is the object assignment of a functor

$$\textbf{\textit{Gauch}} \xrightarrow{\mathcal{F}} \textbf{\textit{Fine}}$$

*and the **Gauch**-arrow η_S is natural for variation of S.*

Observe that in the defintion of a reflection

$$S \xrightarrow{\eta_S} \mathcal{F}S$$

each ***Gauch***-arrow

$$S \xrightarrow{f} A$$

must factor *uniquely* through η_S. This uniqueness can often be seen as a special property of η_S.

5.3 DEFINITION. Let $\mathcal{G}$, as above, be a forgetful functor. Let S be a ***Gauch***-object, and let

$$S \xrightarrow{\eta_S} \mathcal{F}S$$

be a selected ***Gauch***-arrow to a ***Fine***-object $\mathcal{F}S$.

We say η_S is ***Fine***-epic if

$$g \circ \eta_S = h \circ \eta_S \Longrightarrow g = h$$

holds for each parallel pair

$$\begin{tikzcd} \mathcal{F}S \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \end{tikzcd}$$

of ***Fine***-arrows. ■

This terminology would not be acceptable to a hard line category theorist, but the notion is still useful.

5.2 Reflection from ***Meet***

Recall that a $\wedge$-semilattice is a structure

$$(S, \leq, \wedge, \top)$$

where $(S, \leq)$ is a poset with top $\top$ and $\wedge$ is a binary meet on S, that is a binary operation such that

$$z \leq x \wedge y \Longleftrightarrow z \leq x \text{ and } z \leq y$$

for all $x, y, z \in S$. These are the objects of ***Meet***. A morphism

$$S \xrightarrow{f} T$$

between two such object is a function f (from S to T) such that

$$f(\top) = \top \qquad f(x \wedge y) = f(x) \wedge f(y)$$

for all $x, y \in S$. Such a morphism is automatically monotone. These are the arrows of ***Meet***.

Each frame is, can be viewed as, a $\wedge$-semilattice, and each frame morphism is, can be viewed as, a $\wedge$-semilattice morphism. This gives the forgetful functor from ***Frm*** to ***Meet***. Our job in this subsection is to describe the reflection in the other direction.

5.4 DEFINITION. Let S be a poset.

A lower section of S is a subset $L \subseteq S$ such that

$$y \leq x \in L \Longrightarrow y \in L$$

(for $x, y \in S$).

Let $\mathcal{L}S$ be the poset of all lower sections of S under inclusion.

A simple exercise shows that for each subfamily $\mathcal{X} \subseteq \mathcal{L}S$ (where S is a poset) both $\bigcup \mathcal{X}$ and $\bigcap \mathcal{X}$ is a lower section. In particular, $\mathcal{L}S$ is a complete lattice (of a concrete kind). Since $\mathcal{L}S$ sits inside the power set of S another simple calculation shows that $\mathcal{L}S$ is a frame. (In fact, $\mathcal{L}S$ is a topology on S but we don't need that here.)

5.5 DEFINITION. Let S be a poset.

For each $a \in S$

$$\lambda_S(a) = \downarrow a = \{x \in S \mid x \leq a\}$$

is the principal lower section generated by a. ■

This gives a function

$$\lambda_S : S \longrightarrow \mathcal{L}S$$

and, almost trivially, this is monotone. This construction can be carried out for any poset S. When S is a $\wedge$-semilattice we get a bonus.

5.6 LEMMA. *For each $\wedge$-semilattice S the assignment*

$$S \xrightarrow{\lambda_S} \mathcal{L}S$$

is a $\wedge$-semilattice morphism.

Proof. For convenience let $\lambda = \lambda_S$.

Trivially we have $\lambda(\top) = S$ which is the top of $\mathcal{L}S$.

Consider any $a, b \in S$. Then for each $z \in S$ we have

$$\begin{aligned} z \in \lambda(a \wedge b) &\Longleftrightarrow z \leq a \wedge b \\ &\Longleftrightarrow z \leq a \text{ and } z \leq b \\ &\Longleftrightarrow z \in \lambda(a) \text{ and } \lambda(b) \quad \Longleftrightarrow z \in \lambda(a) \cap \lambda(b) \end{aligned}$$

and hence

$$\lambda(a \wedge b) = \lambda(a) \cap \lambda(b)$$

to give the required result. ■

We show that this assignment λ_S give the reflection of the $\wedge$-semilattice S into ***Frm*** in the sense of Definition 5.1. That notion requires a *unique* factorization of certain morphisms f. This is where the notion of Definition 5.3 is useful.

5.7 LEMMA. *For each $\wedge$-semilattice S the $\wedge$-semilattice morphism*

$$S \xrightarrow{\lambda_S} \mathcal{L}S$$

is ***Frm***-*epic, that is*

$$g \circ \lambda_S = h \circ \lambda_S \Longrightarrow g = h$$

holds for each parallel pair

$$\begin{tikzcd} \mathcal{L}S \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \end{tikzcd}$$

of ***Frm***-*arrows.*

Proof. For convenience let $\lambda = \lambda_S$, and suppose

$$g \circ \lambda = h \circ \lambda$$

for some parallel pair g, h of ***Frm***-arrows.

For each $X \in \mathcal{L}S$ we have

$$X = \bigcup\{\downarrow a \mid a \in X\} = \bigcup\{\lambda(a) \mid a \in X\} = \bigcup \lambda^{\rightarrow}(X)$$

and hence

$$g(X) = \bigvee\{(g \circ \lambda)(a) \mid a \in X\} \qquad \bigvee\{(g \circ \lambda)(a) \mid a \in X\} = h(X)$$

since both g and h are $\bigvee$-morphisms. The required result is now immediate. ■

Notice that we have actually proved a little more here. For each poset S the assignment λ_S is ***Sup***-epic, but we don't need that strengthening.

This result gives us 'one half' of the main result of this subsection.

5.8 THEOREM. *For each $\wedge$-semilattice S the morphism*

$$S \xrightarrow{\lambda_S} \mathcal{L}S$$

provides a reflection of S into ***Frm***.

Proof. For convenience let $\lambda = \lambda_S$.

We know that $\mathcal{L}S$ is a frame and, by Lemma 5.6, the assignment λ is a $\wedge$-semilattice morphism. Consider any $\wedge$-semilattice morphism

$$S \xrightarrow{f} A$$

to a frame A. By Lemma 5.7 there is at most one frame morphism

$$\mathcal{L}S \xrightarrow{f^\sharp} A$$

such that $f = f^\sharp \circ \lambda$. Thus it suffices to show that there is at least one such morphism.

For each $X \in \mathcal{L}S$ let

$$f^\sharp(X) = \bigvee f^\to(X) = \bigvee\{f(x) \mid x \in X\}$$

to obtain a function $f^\sharp : \mathcal{L}S \longrightarrow A$. For each $s \in S$ we have

$$f(s) \in f^\to(\downarrow s) \subseteq \downarrow f(s)$$

so that

$$(f^\sharp \circ \lambda)(s) = f^\sharp(\downarrow s) = \bigvee f^\to(\downarrow s) = f(s)$$

to show that $f = f^\sharp \circ \lambda$ (as functions).

It remains to show that $f^\sharp$ is a frame morphism.

To show that $f^\sharp$ passes across arbitrary suprema consider any $\mathcal{X} \subseteq \mathcal{L}S$. We have

$$\begin{aligned} f^\sharp(\bigcup \mathcal{X}) = \bigvee f^\to(\bigcup \mathcal{X}) \qquad\quad &= \bigvee\{f(x) \mid x \in \bigcup \mathcal{X}\} \\ \bigvee f^{\sharp\to}(\mathcal{X}) = \bigvee\{f^\sharp(X) \mid X \in \mathcal{X}\} &= \bigvee\{f(x) \mid x \in X \in \mathcal{X}\} \end{aligned}$$

and hence

$$f^\sharp(\bigcup \mathcal{X}) = \bigvee f^{\sharp\to}(\mathcal{X})$$

as required.

Note that this property ensures that $f^\sharp$ is monotone.

To show that $f^\sharp$ passes across binary infema consider any $X, Y \in \mathcal{L}S$. Since $f^\sharp$ is monotone we have

$$f^\sharp(X \cap Y) \subseteq f^\sharp(X) \wedge f^\sharp(Y)$$

so out final job is to check the converse comparison.

We have

$$f^\sharp(X) \wedge f^\sharp(Y) = \big(\bigvee\{f(x) \mid x \in X\}\big) \wedge \big(\bigvee\{f(y) \mid y \in Y\}\big) = \bigvee\{f(x) \wedge f(y) \mid x \in X, y \in Y\}$$

where the second equality follows by two uses of FDL (in the frame A). Also

$$x \in X, y \in Y \Longrightarrow x \wedge Y \in X \cap Y$$

so that, since f is a $\wedge$-semilattice morphism,

$$f^\sharp(X) \wedge f^\sharp(Y) = \bigvee\{f(x \wedge y) \mid x \in X, y \in Y\} \leq \bigvee\{f(z) \mid z \in X, \cap Y\} = f^\sharp(X \cap Y)$$

as required. ■

Although not directly relevant here, it is worth observing that almost the same construction and proof produces a reflection from ***Pos*** into ***Sup***. You should work through the details of this to see exactly where the frame theoretic properties are used in the proof of Theorem 5.8.

5.3 Reflection from *Dlt*

Recall that a bounded lattice is a structure

$$(D, \leq, \wedge, \top, \vee, \bot)$$

where $(D, \leq)$ is a poset with top $\top$ and bottom $\bot$, and where $\wedge$ and $\vee$ are binary meet and join operations on D, that is

$$\begin{aligned} x \vee y \leq z &\Longleftrightarrow x \leq z \text{ and } y \leq z \\ z \leq x \wedge y &\Longleftrightarrow z \leq x \text{ and } z \leq y \end{aligned}$$

for $x, y, z \in D$. Such a lattice is distributive if both

$$\begin{aligned} &(\forall a, x, y \in D)[a \vee (x \wedge y = (a \vee x) \wedge (a \vee y)] \\ &(\forall a, x, y \in D)[a \wedge (x \vee y = (a \wedge x) \vee (a \wedge y)] \end{aligned}$$

hold. In fact, only one of these identities is necessary since each implies the other.

A d-lattice is a bounded distributive lattice. These are the objects of ***Dlt***.

An arrow

$$D \xrightarrow{\quad f \quad} E$$

of ***Dlt*** is a lattice morphism from D to E. Thus

$$\begin{array}{cc} f(\top) = \top & f(\bot) = \bot \\ f(x \wedge y) = f(x) \wedge f(y) & f(x \vee y) = f(x) \vee f(y) \end{array}$$

for each $x, y \in D$. Such a morphism is automatically monotone.

This category ***Dlt*** sits between two forgetful functors

$$\boldsymbol{Meet} \longleftarrow \boldsymbol{Dlt} \longleftarrow \boldsymbol{Frm}$$

where each forgets both structure and property in various degrees. In this subsection we look at a reflection for the right hand functor.

5.9 DEFINITION. Let D be a d-lattice.

An ideal of D is a non-empty lower section $X \in \mathcal{L}D$ such that

$$x, y \in X \longrightarrow x \vee y \in X$$

for all $x, y \in D$.

Let $\mathcal{I}D$ be the poset of all ideals of D under inclusion. ■

We will show that

$$D \longmapsto \mathcal{I}D$$

is the object assignment of the reflection functor from ***Dlt*** to ***Frm***. Thus our first job is to expose the frame structure of $\mathcal{I}D$.

Almost trivially, for each family $\mathcal{X} \subseteq \mathcal{I}D$ the intersection $\bigcap \mathcal{X}$ is an ideal. Thus the poset $\mathcal{I}D$ has all infima and so is a complete lattice. Observe that the singleton $\{\bot\}$ is an ideal of D, and so is the bottom of $\mathcal{I}D$. Similarly, D is an ideal, and so is the top of $\mathcal{I}D$.

Since $\mathcal{I}D$ has all infima, it also has all suprema. However, these are not just unions.

5.10 LEMMA. *Let D be a d-lattice, let $\mathcal{X} \subseteq \mathcal{I}D$, and let $Y \subseteq D$ be given by*

$$y \in Y \iff (\exists x_1, \dots, x_m \in \bigcup \mathcal{X})[y \leq x_1 \vee \dots \vee x_m]$$

for $y \in D$. Then Y is an ideal and is the supremum of $\mathcal{X}$ in $\mathcal{I}D$.

Proof. Trivially, Y is a non-empty lower section of D. Given $y, z \in Y$ we have

$$y \leq x_1 \vee \dots \vee x_m \qquad z \leq x_{m+1} \vee \dots \vee x_n$$

for $x_1, \dots, x_n \in \bigcup \mathcal{X}$. But now

$$y \vee z \leq x_1 \vee \dots \vee x_n$$

to show that $y \vee z \in Y$. Thus Y is an ideal of D.

Trivially, $\bigcup \mathcal{X} \subseteq Y$, so that $\bigvee \mathcal{X} \subseteq Y$. We need the converse inclusion.

Consider any ideal $Z \in \mathcal{I}D$ with $\bigcup \mathcal{X} \subseteq Z$. Consider any $y \in Y$. We have

$$y \leq x_1 \vee \dots \vee x_m$$

where

$$x_1, \dots, x_m \in \bigcup \mathcal{X} \subseteq Y$$

so that

$$y \leq x_1 \vee \dots \vee x_m \in Z$$

and hence $y \in Z$. Thus $Y \subseteq Z$ and hence $Y = \bigvee \mathcal{X}$. ■

Of course, we need $\mathcal{I}D$ to be a bit more than a complete lattice.

5.11 LEMMA. *For each d-lattice D the poset $\mathcal{I}D$ is a frame.*

Proof. We know that $\mathcal{I}D$ is a complete lattice, so it suffices to show

$$X \cap \bigvee \mathcal{Y} = \bigvee \{X \cap Y \mid Y \in \mathcal{Y}\}$$

for $X \in \mathcal{I}D$ and $\mathcal{Y} \subseteq \mathcal{I}D$. The inclusion '$\supseteq$' is trivial, so it remains to verify the converse inclusion.

Consider

$$z \in X \cap \bigvee \mathcal{Y}$$

so that $z \in X$ and

$$z \leq y_1 \vee \dots \vee y_m$$

where $y_i \in Y_i \in \mathcal{Y}$ for $1 \leq i \leq m$. Since D is distributive we have

$$z = z \wedge (y_1 \vee \dots \vee y_m) = (z \wedge y_1) \vee \dots \vee (z \wedge y_m)$$

where $z \wedge y_i \in X \cap Y_i$ for $1 \leq i \leq m$. This shows that

$$z \in \bigvee \{X \cap Y \mid Y \in \mathcal{Y}\}$$

as required. ■

There is another method of showing that $\mathcal{I}D$ is a frame, namely by exhibiting the implication operation on $\mathcal{I}D$. You will find it instructive to work out the details of this proof.

For each $a \in D$ (a d-lattice) the principal lower section

$$\eta_D(a) = \downarrow a$$

is an ideal of D, that is $\eta(a) \in \mathcal{I}D$.

5.12 LEMMA. *For each d-lattice D the assignment*

$$D \xrightarrow{\eta_D} \mathcal{I}D$$

is a lattice morphism.

Proof. For convenience let $\eta = \eta_D$.

Since

$$\eta(\top) = \downarrow\top = D \qquad \eta(\bot) = \downarrow\bot\{\bot\}$$

it suffices to show

$$\eta(a \wedge b) = \eta(a) \cap \eta(b) \qquad \eta(a \vee b) = \eta(a) \vee \eta(b)$$

for $a, b \in D$. The left hand equality is immediate, as is the inclusion

$$\eta(a \vee b) \subseteq \eta(a) \vee \eta(b)$$

so it suffices to check the converse of this.

Consider $z \in \eta(a) \vee \eta(b)$, that is

$$z \le x \vee y$$

for some $x \le a$ and $y \le b$. Then

$$z \le x \vee y \le a \vee b$$

so that $z \in \eta(a \vee b)$, to give the required result. ■

We will show that the lattice morphism

$$D \xrightarrow{\eta_D} \mathcal{I}D$$

provides a reflection of the d-lattice D into ***Frm***. The proof of this will follow that of Theorem 5.8, so that first thing we need is an analogue of Lemma 5.7 The only minor problem is that suprema in $\mathcal{I}D$ need not be unions.

5.13 LEMMA. *For each d-lattice D the lattice morphism*

$$D \xrightarrow{\eta_D} \mathcal{I}D$$

*is **Frm**-epic, that is*

$$g \circ \eta_S = h \circ \eta_S \Longrightarrow g = h$$

holds for each parallel pair

$$\begin{tikzcd} \mathcal{I}D \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \end{tikzcd}$$

*of **Frm**-arrows.*

Proof. For convenience let $\eta = \eta_S$.
Consider $X \in \mathcal{I}D$ and let

$$\mathcal{X} = \eta^{\rightarrow}(X) = \{\downarrow x \mid x \in X\}$$

to obtain $\mathcal{X} \subseteq \mathcal{I}D$ with $X = \bigcup \mathcal{X}$. Consider $z \in \bigvee \mathcal{X}$ (as computed in $\mathcal{I}D$). Then

$$z \leq x_1 \vee \cdots \vee x_m = y$$

for some $x_1, \ldots, x_m \in X$. But now $y \in X$ (since X is an ideal) so that $z \in \downarrow y \in \mathcal{X}$. This shows that

$$\bigvee \mathcal{X} = \bigcup \mathcal{X} = X$$

which helps us out of the minor problem
Suppose

$$g \circ \eta = h \circ \eta$$

for some parallel pair g, h of ***Frm***-arrows. Then, since both g and h are $\bigvee$-morphisms we have

$$g(X) = g(\bigvee \mathcal{X}) = \bigvee (g \circ \eta)^{\rightarrow}(X) = \bigvee (h \circ \eta)^{\rightarrow}(X) = h(\bigvee \mathcal{X}) = h(X)$$

to give the required result. ■

With this we can obtain the reflection result. This is the analogue of Theorem 5.8, and much of the proof is exactly the same.

5.14 THEOREM. *For each d-lattice D the morphism*

$$D \xrightarrow{\eta_D} \mathcal{I}D$$

*provides a reflection of D into **Frm**.*

Proof. For convenience let $\eta = \eta_D$.
We know that $\mathcal{I}D$ is a frame and, by Lemma 5.12, the assignment η is a lattice morphism. Consider any lattice morphism

$$D \xrightarrow{f} A$$

to a frame A. By Lemma 5.13 there is at most one frame morphism

$$\mathcal{I}D \xrightarrow{f^\sharp} A$$

such that $f = f^\sharp \circ \eta$. Thus it suffices to show that there is at least one such morphism.

For each $X \in \mathcal{I}D$ let

$$f^\sharp(X) = \bigvee f^\rightarrow(X) = \bigvee\{f(x) \mid x \in X\}$$

to obtain a function $f^\sharp : \mathcal{L}S \longrightarrow A$. As in the proof of Theorem 5.8 it suffices to show that $f^\sharp$ is a frame morphism. The proof that $f^\sharp$ passes across binary meets is exactly the same as in the proof of Theorem 5.8. Thus it suffices to show that $f^\sharp$ passes across arbitrary suprema.

Consider $\mathcal{X} \subseteq \mathcal{I}D$. We must show that

$$LHS = f^\sharp(\bigvee\mathcal{X}) \qquad RHS = \bigvee f^{\sharp\rightarrow}(\mathcal{X})$$

are equal. However, we have

$$\begin{aligned} LHS &= \bigvee f^\rightarrow(\bigvee\mathcal{X}) &&= \bigvee\{f(y) \mid y \in \bigvee\mathcal{X}\} \\ RHS &= \bigvee\{f^\sharp(X) \mid X \in \mathcal{X}\} &&= \bigvee\{f(y) \mid x \in X \in \bigvee\mathcal{X}\} \end{aligned}$$

so that $RHS \leq LHS$ is immediate. Conversely, consider any $y \in \bigvee\mathcal{X}$. We have

$$y \leq x_1 \vee \cdots \vee x_m$$

where $x_i \in X_i \in \mathcal{X}$ for $1 \leq i \leq m$. Thus, since f is a lattice morphism, we have

$$f(y) \leq f(x_1 \vee \cdots \vee x_m) = f(x_1) \vee \cdots \vee f(x_m) \leq RHS$$

and hence

$$LHS = \bigvee\{f(y) \mid y \in \bigvee\mathcal{X}\} \leq RHS$$

as required. ■

This deals with the reflection from ***Dlt*** to ***Frm***. Before we move on it is worth making a few remarks about the reflection from ***Meet*** to ***Dlt***. This can be set up using a construction very simular to that used in Subsection 5.2. The main difference is that the target here is an object with only finitary suprema, whereas there it has arbitarry suprema. It is an instructive exercise to worh out how this is done, and to check that the composite of the two reflections

$$\textbf{\textit{Meet}} \longrightarrow \textbf{\textit{Dlt}} \longrightarrow \textbf{\textit{Frm}}$$

is essentially the same as that of Subsection 5.2.

We conclude this subsection with a result which at fist sight looks to be little more than an interesting curiosity. However, it is more than that.

Consider a frame A. This is a d-lattice, and so has its own associated frame $\mathcal{I}d\,A$ of ideals. We have a d-lattice embedding

$$A \xrightarrow{\eta_A} \mathcal{I}d\,A$$

but in general this is not a frame embedding.

Each ideal I of A is a subset of A, and so has a supremum $\bigvee I$ in A. This gives us an assignment

$$\begin{array}{rcl} \mathcal{I}d\,A & \xrightarrow{\zeta_A} & A \\ I & \longmapsto & \bigvee I \end{array}$$

where

$$\zeta_A \circ \eta_A = \boldsymbol{id}_A$$

(since each value of η_A is a principal ideal).

5.15 THEOREM. *For each frame A he assignment*

$$\mathcal{I}d\,A \xrightarrow{\zeta_A} A$$

is a frame morphism.

Proof. For convenience let $\zeta = \zeta_A$.

Trivially we have

$$\zeta(\{\bot\}) = \bot \qquad s\zeta(A) = \top$$

and ζ is monotone. Thus it suiffices to show

$$\zeta(I) \wedge \zeta(J) \leq \zeta(I \cap J) \qquad \zeta(\bigvee \mathcal{J}) \leq \bigvee \{\zeta(J) \mid J \in \mathcal{J}\}$$

for each $I, J \in \mathcal{I}d\,A$ and $\mathcal{J} \subseteq \mathcal{I}d\,A$.

For the left hand comparison we have

$$\zeta(I) \wedge \zeta(J) = (\bigvee I) \wedge (\bigvee J) = \bigvee \{a \wedge b \mid a \in I, b \in J\} \leq \bigvee (I \cap J) = \zeta(I \cap J)$$

where the second, crucial, equality follows by two uses of FDL.

For the right hand comparison consider any $b \in \bigvee \mathcal{J}$. We have

$$b \leq a_1 \vee \cdots a_m$$

for some selection

$$a_1 \in J_1 \in \mathcal{J}, \ldots, a_m \in J_m \in \mathcal{J}$$

of ideals from $\mathcal{J}$ and elements from these ideals. For each such b we have

$$b \leq \bigvee J_1 \vee \cdots \vee \bigvee J_m \leq \bigvee \mathcal{J}$$

to give the rquired result. ■

Sometimes the map ζ_A is called the structure morphism of A.

5.4 Reflection from *Sup*

When I was preparing these notes I set up the various sections and subsections as listed in the Contents. When I came to write this subsection I realized that I didn't know how to reflect a $\bigvee$-semilattice into a frame. Thus this is a very short subsection.

If I find out more information on this aspect, then I will let you know.

5.5 Reflection from *Set*

Each frame can be viewed as a set to give us a forgetful functor

$$\boldsymbol{Set} \longleftarrow \boldsymbol{Frm}$$

to the category of sets and functions. In this subsection we look at the left adjoint

$$\boldsymbol{Set} \xrightarrow{\Phi} \boldsymbol{Frm}$$

to this forgetful functor. Thus to each set X we attach a frame ΦX which is often called the free frame generated by X. This last notion is a bit finicky to make precise, but it is intuitively clear what the phrase means.

We give three versions of the reflector Φ.

In Block 5.5.1 we show that Φ can be obtained as the composite of two functors, each of which is a reflector. The two components are easy to describe, and one of them is the reflector of Subsection 5.3. The required universal property of Φ is an easy consequence of the corresponding properties of the two components. However, sometimes we need to get inside Φ, and this particular description does help us do that.

In Block 5.5.2 we give a direct construction of Φ, and verify the required properties by direct calculation. This construction is essentially the same as that of Block 5.5.1 but with most of the categorical aspects stripped away. There is nothing wrong with the construction, but it can leave you a bit disappointed. It doesn't really explain why the reflection exists.

In Block 5.5.3 we give another direct construction of Φ which really gets to the heart of the matter. Most of the required verifications are simple, one line, affairs. And right in the middle there is a simple result which, as we will see, makes the whole construction work. However, it has to be said that at this stage the whole story can not be told. There are no gaps in the proof, but there is other relevant information that is not given, because we don't have the background material. That will be done in [5].

5.5.1 A 2-step frame reflection

For the first construction of the reflector

$$\boldsymbol{Set} \xrightarrow{\Phi} \boldsymbol{Frm}$$

we consider the two forgetful functors

$$\boldsymbol{Set} \longleftarrow \boldsymbol{Meet} \longleftarrow \boldsymbol{Frm}$$

where ***Meet*** is the category of $\wedge$-semilattices. By Subsection 5.3 we know the right hand functor has a reflector, and in this subsection we show that the left hand functor has a reflector. Thus we obtain a pair of functors

$$\boldsymbol{Set} \xrightarrow{\mathcal{M}} \boldsymbol{Meet} \xrightarrow{\mathcal{L}} \boldsymbol{Frm}$$

together with natural assignments

$$X \xrightarrow{\mu_X} \mathcal{M}X \qquad\qquad M \xrightarrow{\lambda_M} \mathcal{L}M$$

for each set X and each $\wedge$-semilattice M. These two assignments have certain universal properties. We wish to show that

$$\Phi = \mathcal{L} \circ \mathcal{M}$$

is the required reflector. But this is a simple peace of abstract nonsense.

5.16 THEOREM. *For each set X the assignment*

$$X \xrightarrow{\eta_X = \lambda_{\mathcal{M}X} \circ \mu_X} \Phi X$$

reflects X into a frame.

Proof. We know that

$$\Phi X = \mathcal{L}(\mathcal{M}X)$$

is a frame, so it suffices to verify the required universal property.

To this end consider any function

$$X \xrightarrow{f} A$$

from X to a frame A. Since A is a $\wedge$-semillatice we obtain a commuting triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mu_X"'] & & A \\ & \mathcal{M}X \arrow[ur, "f^\natural"'] & \end{tikzcd}$$

for some unique $\wedge$-semilattice morphism $f^\natural$. Using this we obtain a second commuting triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mu_X"'] & & A & \\ & \mathcal{M}X \arrow[ur, "f^\natural"] \arrow[rr, "\lambda_{\mathcal{M}X}"'] & & \Phi X \arrow[ul, "f^\sharp"'] \end{tikzcd}$$

for some unique frame morphism $f^\sharp$. This, mor or less, completes the proof. ■

Our problem now is to produce the reflector $\mathcal{M}$. This is just as easy to describe, but there is a little twist that can cause a hiccough. (The same method we use will also produce a reflection from the category of posets into ***Meet***. It is instructive to go through that construction because the role of the twist becomes more obvious.)

Let X be any set and let

$$\mathcal{P}_{cof} X$$

be the poset of cofinite subsets of X. These are the subsets of the form

$$F' = X - F$$

for finite $F \subseteq X$. Since

$$F' \cap G' = (F \cup G)'$$

we see that $\mathcal{P}_{cof}X$ is a $\wedge$-semilattice with top $X = \emptyset'$. We show that the assignment

$$\begin{array}{ccc} X & \xrightarrow{\mu_X} & \mathcal{P}_{cof}X \\ x & \longmapsto & \{x\}' \end{array}$$

reflects X into ***Meet***.

We deal with the required uniqueness in the standard way.

5.17 LEMMA. *For each set X the assignment*

$$X \xrightarrow{\mu_X} \mathcal{P}_{cof}X$$

*is **Meet**-epic.*

Proof. For convenience let $\mu = \mu_X$.

Consider any parallel pair

$$\begin{tikzcd} \mathcal{P}_{cof}X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & M \end{tikzcd}$$

of functions to a $\wedge$-semilattice, and suppose $g \circ \mu = h \circ \mu$. Consider an arbitrary member F' of $\mathcal{P}_{cof}X$. We have

$$F = \bigcup \{\{x\} \mid x \in F\}$$

and this is a union of finitely many singletons. Thus

$$F' = \bigcap \{\{x\}' \mid x \in F\} = \bigcap \{\mu(x) \mid x \in F\}$$

and this is an intersection of finitely many singletons. The presevation property of g and h now gives

$$\begin{aligned} g(F') &= g(\bigcap \{\mu(x) \mid x \in F\}) = \bigwedge \{g(\mu(x)) \mid x \in F\} \\ h(F') &= h(\bigcap \{\mu(x) \mid x \in F\}) = \bigwedge \{h(\mu(x)) \mid x \in F\} \end{aligned}$$

which leads to the required result. ∎

The whole proof is just as easy.

5.18 THEOREM. *For each function*

$$X \xrightarrow{f} M$$

*from a set X to a $\wedge$-semilattice M, there is a unique **Meet** morphism*

$$\mathcal{P}_{cof}X \xrightarrow{f^\sharp} M$$

such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mu_X"'] & & M \\ & \mathcal{P}_{cof}X \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

commutes. This morphism $f^\sharp$ is given by

$$f^\sharp(F') = \bigwedge f^{\rightarrow}(F)$$

for each finite $F \subseteq X$.

Proof. Much of this is routine. The only real problem is to show that $f^\sharp$ is a $\wedge$-morphism, that is

$$f^\sharp(F' \cap G') = f^\sharp(F) \wedge f^\sharp(G')$$

for finite $F, G \subseteq S$. But

$$f^\sharp(F' \cap G') = f^\sharp((F \cup G)') = \bigwedge\{f(z) \mid z \in F \cup G\}$$

and

$$f^\sharp(F') \wedge f^\sharp(G') = \bigwedge\{f(x) \mid x \in F\} \wedge \bigwedge\{f(y) \mid y \in G\}$$

so the required equality is immediate. ■

These two functors

$$\boldsymbol{Set} \xrightarrow{\mathcal{M}} \boldsymbol{Meet} \xrightarrow{\mathcal{L}} \boldsymbol{Frm}$$

combine to produce the reflector of ***Set*** into ***Frm***. Let's see what this does.

Consider any set X. At the first step we take the semilattice

$$\mathcal{P}_{cof}X$$

of cofinite subsets of S. We then take the frame of lower sections of this poset. Thus a typical member of the constructed frame has the form

$$\{F' \mid F \in \mathcal{F}\}$$

where $\mathcal{F}$ is certain family of finite subsets of X. Because we want to obtain a lower section of $\mathcal{P}_{cof}X$, this family $\mathcal{F}$ must be upwards closed, that is we require

$$F \in \mathcal{F} \Longrightarrow G \in \mathcal{F}$$

for all *finite* subsets $F \subseteq G$ of X.

In short, for each set X the reflection of x into ***Frm*** is carried by the upwards closed subfamilies of $\mathcal{P}_{fin}X$, the family of finite subsets of X.

This is the carrier of the required reflection of X. I will leave you to sort out the carried lattice theoretic attributes for yourself. Perhaps you can already see that this description is a bit convoluted, and not very helpful if we have to do any particular calculations. Luckily, there is a neater description of the reflector.

5.5.2 A 1-step frame reflection

In this and the next block we describe a 1-step reflection functor

$$\boldsymbol{Set} \xrightarrow{\Phi} \boldsymbol{Frm}$$

which does not go via an intermediate category. The two blocks both produce the same functor. The difference is that in this block we give a more 'elementary' description of the functor and get involved in one or two messy details, whereas in the next block we try to expose the underlying categorical reasons for the existence of the reflector. You my prefer to skip this block and go straight to the next one.

Recall that sierpinski space

$$\mathsf{2} = \{0, 1\} \qquad \mathcal{O}\mathsf{2} = \{\emptyset, \{1\}, \mathsf{2}\}$$

is the two point space with one point open and the other not. It is convenient to label these points as indicated. In other words, sierpinski space is the two element poset $0 < 1$ furnished with the Alexandroff topology.

For each set X we have the set

$$\boldsymbol{Set}[X, \mathsf{2}]$$

of function

$$p : S \longrightarrow \mathsf{2}$$

from X to $\mathsf{2}$. Sometimes these set of functions is written as ${}^X\mathsf{2}$ or as $\mathsf{2}^X$. Later you will see why we have written it as $\boldsymbol{Set}[X, \mathsf{2}]$. Notice that the members p of $\boldsymbol{Set}[X, \mathsf{2}]$ are just the characteristic functions of subsets of X. For that reason we refer to each such p as a $\boldsymbol{Set}$-character of X, or just character when there is little chance of confusion. (Later we will meet $\boldsymbol{Alg}$-characters for several categories $\boldsymbol{Alg}$ of algebras.)

Since $\mathsf{2}$ is a topological space we can use the product topology to obtain a topology

$$\Phi X = \mathcal{O}\boldsymbol{Set}[X, \mathsf{2}]$$

on this set of characters. In due course we show that ΦX is the free frame generated by X. To do that we need a bit of notation.

We work with

finite subsets of X	families of finite subsets of X
$F \in \mathcal{P}_{fin}X$	$\mathcal{F} \subseteq \mathcal{P}_{fin}X$

and it is convenient to use this fount convention to help us distinguish between these.

5.19 DEFINITION. Let X be an arbitrary set.

(a) For each $x \in F$ we let

$$p \in \eta(x) \Longleftrightarrow p(x) = 1$$

for $p \in \boldsymbol{Set}[X, \mathsf{2}]$ to produce a subset $\eta(x)$ of $\boldsymbol{Set}[X, \mathsf{2}]$.

(b) For each $F \in \mathcal{P}_{fin}X$ we let

$$\eta(F) = \bigcap \{\eta(x) \mid x \in F\}$$

to produce a subset $\eta(F)$ of $\boldsymbol{Set}[X, 2]$.

(c) For each $\mathcal{F} \subseteq \mathcal{P}_{fin}X$ we let

$$\eta\langle\mathcal{F}\rangle = \bigcup\{\eta(F) \mid F \in \mathcal{F}\}$$

to produce a subset $\eta\langle\mathcal{F}\rangle$ of $\boldsymbol{Set}[X, 2]$. ■

In other words, $\eta(x)$ is a typical subbasic open set of $\boldsymbol{Set}[X, 2]$. Similarly, $\eta(F)$ is a typical basic open set of $\boldsymbol{Set}[X, 2]$, and $\eta\langle\mathcal{F}\rangle$ is a typical open set of $\boldsymbol{Set}[X, 2]$. Observe that

$$\eta(F) \cap \eta(G) = \eta(F \cup G)$$

for $F, G \in \mathcal{P}_{fin}X$.

We will show that the assignment

$$X \xrightarrow{\ \eta\ } \Phi X = \mathcal{O}\boldsymbol{Set}[X, 2]$$

reflects S into $\boldsymbol{Frm}$. Strictly speaking, we should have written η_X for η, to indicate its parent set. We don't need to do that in this block, but we will in the next block.

Reflection are concerned with unique factorization of certain morphisms. To achieve that uniqueness here we prove an analogue of Lemmas 5.7 and 5.13

5.20 LEMMA. *For each set X the function*

$$X \xrightarrow{\ \eta\ } \Phi X$$

is $\boldsymbol{Frm}$-epic, that is

$$g \circ \eta = h \circ \eta \Longrightarrow g = h$$

holds for each parallel pair

$$\Phi X \underset{h}{\overset{g}{\rightrightarrows}} A$$

of $\boldsymbol{Frm}$-arrows.

Proof. Suppose

$$g \circ \eta = h \circ \eta$$

for a pair of frame morphisms g, h, as indicate. In other words, we have

$$g(W) = h(W)$$

for each subbasic open subset $W \in \mathcal{O}\boldsymbol{Set}[X, 2]$, since each such W has the form $\eta(x)$ for some $x \in X$.

Consider any basic open set V of $\mathcal{O}\boldsymbol{Set}[X, 2]$. This has the form $\eta(F)$ for some $F \in \mathcal{P}_{fin}X$, and so is a finite intersection of subbasic open sets. Thus

$$g(V) = h(V)$$

since both g and h are $\wedge$-semilattice morphisms.

Consider any open set U of $\mathcal{O}\boldsymbol{Set}[X,2]$. This has the form $\eta\langle\mathcal{F}\rangle$ for some $\mathcal{F}\subseteq\mathcal{P}_{\mathit{fin}}X$, and so is a union of basic open sets. Thus

$$g(U)=h(U)$$

since both g and h are $\bigvee$-morphisms. ■

Now come the substantial part of the proof.

Given a function

$$X\xrightarrow{f}A$$

to some frame A, we must produce a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\eta"'] & & A \\ & \Phi X \arrow[ur, "f^\sharp"'] & \end{tikzcd}$$

for some frame morphism $f^\sharp$. By Lemma 5.20 there is at most one such morphism, and we have to exhibit it. What can it be? Consider any $U\in\Phi X$. This has the form

$$U=\eta\langle\mathcal{F}\rangle=\bigcup\{\eta(F)\mid F\in\mathcal{F}\}$$

for some $\mathcal{F}\subseteq\mathcal{P}_{\mathit{fin}}X$. Thus

$$f^\sharp(U)=\bigvee\{f^\sharp(\eta(F))\mid F\in\mathcal{F}\}$$

since $f^\sharp$ passes across suprema. For each $F\in\mathcal{F}$ we have

$$F=\{x_1,\ldots,x_m\}$$

(for some $x_1,\ldots,x_m\in S$), and then

$$\eta(F)=\eta(x_1)\cap\cdots\cap\eta(x_m)$$

to give

$$f^\sharp(\eta(F))=(f^\sharp\circ\eta)(x_1)\wedge\cdots\wedge(f^\sharp\circ\eta)(x_m)$$

since $f^\sharp$ passes across meets. But

$$f^\sharp\circ\eta=f$$

so that

$$f^\sharp(\eta(F))=\bigwedge f^{\rightarrow}(F)$$

and hence

$$(\sharp)\quad \text{if } U=\eta\langle\mathcal{F}\rangle \text{ then } f^\sharp(U)=\bigvee\{\bigwedge f^{\rightarrow}(F)\mid F\in\mathcal{F}\}$$

which appears to be an explicit description of $f^\sharp$.

We would like to use $(\sharp)$ as a definition of this fill-in. Unfortunately, this description depends on the family $\mathcal{F}$ with $U=\eta\langle\mathcal{F}\rangle$. For a particular $U\in\mathcal{O}\boldsymbol{Set}[X,2]$ there may be many such families $\mathcal{F}$. Thus before we can use $(\sharp)$ as a definition we must show that the right hand side is independent of the choice of $\mathcal{X}$. To achieve that we prove something slightly stronger.

5.21 LEMMA. *For the situation described above, suppose*

$$\eta(F) \subseteq \eta\langle \mathcal{G} \rangle$$

for $F \in \mathcal{P}_{fin}x$ and $\mathcal{G} \subseteq \mathcal{P}_{fin}X$. Then

$$\bigwedge f^{\rightarrow}(F) \leq \bigvee\{\bigwedge f^{\rightarrow}(G) \mid G \in \mathcal{G}\}$$

holds.

Proof. For convenience let

$$a = \bigwedge f^{\rightarrow}(F) \qquad b = \bigvee\{\bigwedge f^{\rightarrow}(G) \mid G \in \mathcal{G}\}$$

so we require $a \leq b$.

Consider the function

$$p : S \longrightarrow 2$$

where

$$p(z) = 1 \iff a \leq f(z)$$

for $z \in X$. For each $x \in F$ we have $a \leq f(x)$ and hence $p(x) = 1$. Thus

$$p \in \eta(F) \subseteq \eta\langle \mathcal{G} \rangle$$

so that $p \in \eta(Y)$ for some $Y \in \mathcal{Y}$. But now $p(y) = 1$ and hence $a \leq f(y)$ for each $y \in G$, to give

$$a \leq \bigwedge f^{\rightarrow}(G) \leq b$$

for the required result. ■

A particular case of this shows that the value $f^{\sharp}(U)$ given by ($\sharp$) is independent of the representation of U.

5.22 COROLLARY. *For each $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}_{fin}X$, if*

$$\eta\langle \mathcal{F} \rangle = \eta\langle \mathcal{G} \rangle$$

then

$$\bigvee\{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}\} = \bigvee\{\bigwedge f^{\rightarrow}(G) \mid G \in \mathcal{G}\}$$

holds.

This result shows that ($\sharp$) is a well-defined description of a function

$$f^{\sharp} : \Phi X \longrightarrow A$$

from the 'free' topology to the target frame. Our job now is to show that this function has all the required properties. We do this bit by bit.

5.23 LEMMA. *The function $f^{\sharp}$ is monotone.*

Proof. Consider open sets $U \subseteq V$ of $\boldsymbol{Set}[X, 2]$. Let

$$U = \eta\langle\mathcal{F}\rangle \qquad V = \eta\langle\mathcal{G}\rangle$$

for $\mathcal{F}, \mathcal{G} \in \mathcal{P}_{\mathit{fin}} x$. For each $F \in \mathcal{F}$ we have

$$\eta(F) \subseteq U \subseteq V = \eta\langle\mathcal{G}\rangle$$

and hence

$$\bigwedge f^{\rightarrow}(F) \leq \bigvee\{\bigwedge f^{\rightarrow}(G) \mid G \in \mathcal{G}\} = f^{\sharp}(V)$$

by Lemma 5.21. But now

$$f^{\sharp}(U) = \bigvee\{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}\} \leq f^{\sharp}(V)$$

to give the required result. ■

Next we look at half the required morphism properties.

5.24 LEMMA. *The function $f^{\sharp}$ is a $\wedge$-morphism.*

Proof. The top T of ΦX is the whole space $\boldsymbol{Set}[X, 2]$. To show

$$f^{\sharp}(T) = \top_A$$

we need to locate $\mathcal{F} \subseteq \mathcal{P}_{\mathit{fin}} X$ with $T = \eta\langle\mathcal{F}\rangle$. We show that

$$\mathcal{F} = \{\emptyset\}$$

will do where $\emptyset \subseteq \mathcal{P}_{\mathit{fin}} X$. We have

$$\eta(\emptyset) = \bigcap \emptyset = X$$

for remember that this intersection is computed in ΦX. Thus

$$\eta\langle\{\emptyset\}\rangle = \bigcup\{\eta(F) \mid F \in \{\emptyset\}\} = \eta(\emptyset) = X$$

as we claimed. Similarly since $f^{\rightarrow}(\emptyset)$ is empty we have

$$\bigwedge f^{\rightarrow}(\emptyset) = \top_A$$

for remember that this infimum is computed in A. Thus

$$f^{\sharp}(T) = \sup\{\bigwedge f^{\rightarrow}(F) \mid F \in \{\emptyset\}\} = \bigwedge f^{\rightarrow}(\emptyset) = \top_A$$

to show that $f^{\sharp}$ preserves top.

For the more substantial requirement consider $U, V \in \Phi X$ and let

$$U = \eta\langle\mathcal{F}\rangle \qquad V = \eta\langle\mathcal{G}\rangle$$

for $\mathcal{F}, \mathcal{G} \subseteq \mathcal{P}_{\mathit{fin}} X$. Let $\mathcal{H}$ be the family of all sets $F \cup G$ for $F \in \mathcal{F}$ and $G \in \mathcal{G}$. We have

$$\begin{aligned} U \cap V &= \bigcup\{\eta(F) \mid F \in \mathcal{F}\} \cap \bigcup\{\eta(G) \mid G \in \mathcal{G}\} \\ &= \bigcup\{\eta(F) \cap \eta(G) \mid F \in \mathcal{F}, G \in \mathcal{G}\} \\ &= \bigcup\{\eta(F \cup G) \mid F \in \mathcal{F}, G \in \mathcal{G}\} \\ &= \bigcup\{\eta(H) \mid H \in \mathcal{H}\} \qquad\qquad = \eta\langle\mathcal{H}\rangle \end{aligned}$$

to give a representation of $U \cap V$. Remember also that

$$\bigwedge f^{\rightarrow}(F) \wedge \bigwedge f^{\rightarrow}(G) = \bigwedge f^{\rightarrow}(F \cup H)$$

for arbitrary X, Y. With these we have

$$\begin{aligned} f^{\sharp}(U) &= \bigvee\{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}\} \\ f^{\sharp}(V) &= \bigvee\{\bigwedge f^{\rightarrow}(G) \mid G \in \mathcal{G}\} \\ f^{\sharp}(U \cap V) &= \bigvee\{\bigwedge f^{\rightarrow}(H) \mid H \in \mathcal{H}\} \end{aligned}$$

so that two uses of FDL (on A) gives

$$\begin{aligned} f^{\sharp}(U) \wedge f^{\sharp}(V) &= \bigvee\{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}\} \wedge \bigvee\{\bigwedge f^{\rightarrow}(G) \mid G \in \mathcal{G}\} \\ &= \bigvee\{\bigwedge f^{\rightarrow}(F) \wedge \bigwedge f^{\rightarrow}(G) \mid F \in \mathcal{F}, G \in \mathcal{G}\} \\ &= \bigvee\{\bigwedge f^{\rightarrow}(F \cup G) \mid F \in \mathcal{F}, G \in \mathcal{G}\} \\ &= \bigvee\{\bigwedge f^{\rightarrow}(H) \mid H \in \mathcal{H}\} \qquad = f^{\sharp}(U \cap V) \end{aligned}$$

as required. ■

There is just one more step to go.

5.25 LEMMA. *The function $f^{\sharp}$ is a frame morphism.*

Proof. By Lemmas 5.23 and 5.24 it suffices to show that $f^{\sharp}$ is a $\bigvee$-morphism.
The bottom of ΦX is the empty set $\emptyset$ (as a subset of X). To show

$$f^{\sharp}(\emptyset) = \bot_A$$

we need to locate $\mathcal{F} \subseteq \mathcal{P}_{\mathit{fin}} X$ with $\emptyset = \eta\langle\mathcal{F}\rangle$. In fact $\mathcal{F} = \emptyset$ will do since

$$\eta\langle\emptyset\rangle = \bigcup\{\eta(F) \mid F \in \emptyset\} = \bigcup\emptyset = \emptyset$$

for remember that this union is computed in ΦX. Similarly since $f^{\rightarrow}(\emptyset)$ is empty we have

$$\bigwedge f^{\rightarrow}(\emptyset) = \top_A$$

for remember that this infimum is computed in A. Using this we have

$$f^{\sharp}(\emptyset) = \bigvee\{\bigwedge f^{\rightarrow}(F) \mid F \in \emptyset\} = \bigvee\emptyset = \bot_A$$

for remember that this supremum is computed in $\mathcal{P}X$.
For the more substantial requirement it suffices to show that

$$f^{\sharp}(\bigcup\mathcal{U}) \leq \{f^{\sharp}(U) \mid U \in \mathcal{U}\}$$

for $\mathcal{U} \subseteq \Phi S$.
Consider such a $\mathcal{U}$. We may index this as

$$\mathcal{U} = \{U_i \mid i \in I\}$$

for some index set I. For each index i let

$$U_i = \eta\langle \mathcal{F}_i \rangle$$

for some $\mathcal{F}_i \subseteq \mathcal{P}_{\mathit{fin}} X$. Let

$$\mathcal{F} = \bigcup \{\mathcal{F}_i \mid i \in I\}$$

to obtain another $\mathcal{F} \subseteq \mathcal{P}_{\mathit{fin}} X$. We have

$$\begin{aligned}
\bigcup \mathcal{U} &= \bigcup \{U_i \mid i \in I\} \\
&= \bigcup \{\eta\langle \mathcal{F}_i \rangle \mid i \in I\} \\
&= \bigcup \{\bigcup \{\eta(F) \mid F \in \mathcal{F}_i\} \mid i \in I\} \\
&= \bigcup \{\eta(F) \mid F \in \mathcal{F}_i, i \in I\} \\
&= \bigcup \{\eta(F) \mid F \in \mathcal{F}\} \qquad\qquad = \eta\langle \mathcal{X} \rangle
\end{aligned}$$

to give a representation of $\bigcup \mathcal{U}$. Using this we have

$$f^\sharp(U_i) = \bigvee \{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}_i\}$$

and

$$f^\sharp(\bigcup \mathcal{U}) = \bigvee \{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}\}$$

where this second supremum can be partitioned using I. This gives

$$f^\sharp(\bigcup \mathcal{U}) = \bigvee \{f^\sharp(U_i) \mid i \in I\}$$

as required ■

This almost completes the proof of the following.

5.26 THEOREM. *For each function*

$$X \xrightarrow{\ f\ } A$$

from a set X to a frame A, there is a unique frame morphism

$$\Phi X \xrightarrow{\ f^\sharp\ } A$$

such that

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\eta"'] & & A \\
& \Phi X \arrow[ur, "f^\sharp"'] &
\end{tikzcd}$$

commutes. This morphism $f^\sharp$ is given by ($\sharp$).

Proof. The only thing left to do is to check that $f^\sharp \circ \eta = f$. To this end consider any $x \in X$, let

$$F = \{x\} \qquad \mathcal{F} = \{F\}$$

so that

$$\eta(F) = \eta(x) \qquad \eta\langle\mathcal{F}\rangle = \eta(F) = \eta(x)$$

and hence

$$(f^\sharp \circ \eta)(x) = \bigvee\{\bigwedge f^{\rightarrow}(F) \mid F \in \mathcal{F}\} = f(x)$$

as required. ∎

The fact that this reflection ΦX is the topology on a rather canonical space $\boldsymbol{Set}[X, 2]$ associated with the set X will be useful later.

5.5.3 A better 1-step relection

The idea of the previous block seems quite simple, but some of the calculations do tend to get to obscure what is going on. There is a cleaner account of the construction using various functorial properties. At the heart of the construction is the relationship between a frame and its spectrum as a d-lattice. That material is developed in [5]. Nevertheless, with a bit of give and take, it is possible to give a version of this cleaner account.

Consider a frame A. We may forget all of its structure and view it as a set to produce a topology ΦA and an assignment

$$A \xrightarrow{\eta_A} \Phi A$$

which is just a function. We may also view A as a d-lattice and so obtain the associated frame $\mathcal{I}d\,A$ of ideals of A.

5.27 THEOREM. *For each frame A there is a unique frame morphism*

$$\Phi A \xrightarrow{\iota_A} \mathcal{I}d\,A$$

and furthermore

$$(\iota_A \circ \eta_A)(a) = {\downarrow} a$$

for each $a \in A$.

We will prove this result at the end of this block. Before that let's see how it can be used to produce the required reflection.

We use a slight modification of ι_A.

By Theorem 5.15 we have a frame morphism

$$\begin{array}{rcl} \mathcal{I}d\,A & \xrightarrow{\zeta_A} & A \\ I & \longmapsto & \bigvee I \end{array}$$

which sends each ideal I of A to its supremum in A. In particular, if I is principal then this assignment picks out the principal generator. Let

$$\Phi A \xrightarrow{\delta_A} A$$

be the composite

$$\Phi A \xrightarrow{\iota_A} \mathcal{I}d\, A \xrightarrow{\zeta_A} A$$

of the two frame morphism. We have

$$\delta_A \circ \eta_A = \boldsymbol{id}_A$$

since each value of $\iota_A \circ \eta_A$ is principal.

Consider a function

$$X \xrightarrow{f} A$$

from a set X to a frame A. How might we produce a factorization

$$f = f^\sharp \circ \eta_X$$

of f where $f^\sharp$ is a frame morphism?

The function induces a function

$$\begin{array}{ccc} \boldsymbol{Set}[X,2] & \xleftarrow{\phi} & \boldsymbol{Set}[A,2] \\ p \circ f & \longleftarrow\!\!\dashv & p \end{array}$$

in the opposite direction. We easily check that ϕ is continuous with respect to

$$\Phi X \qquad\qquad \Phi A$$

the two carried topologies. In fact

$$\phi^{\leftarrow}(\eta_X(x)) = \eta_A(f(x))$$

for each $x \in X$. Thus we obtain a commuting square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & A \arrow[d, "\eta_A"] \\ \Phi X \arrow[r, "\phi^{\leftarrow}"'] & \Phi A \end{tikzcd}$$

of functions where $\phi^{\leftarrow}$ is a frame morphism. With

$$f^\sharp = \delta_A \circ \phi^{\leftarrow}$$

we have

$$f^\sharp \circ \eta_X = \delta \circ \phi^{\leftarrow} \circ \eta_X = \delta \circ \eta_A \circ f = f$$

for the required factorization.

This shows that the morphism ι_A is the heart of the reflection property. Of course, since η_A is $\boldsymbol{Frm}$-epic, there is at most one such morphism ι_A. Thus to prove Theorem 5.27 it suffices to exhibit one such morphism.

We now deal entirely with the frame A. Thus we may drop various decorations and write

$$\eta = \eta_A \qquad \iota = \iota_A$$

for the known function and the required morphism.

Let

$$T = \boldsymbol{Set}[A, 2] \qquad S = \boldsymbol{Dlt}[A, 2]$$

so that T is the set of functions which carries the topology

$$\Phi A = \mathcal{O}T$$

and S is a certain subsets of T. In fact, S is the set of ***Dlt***-characters of A, the set of those functions

$$p : A \longrightarrow 2$$

which happen to be ***Dlt***-morphisms, that is

$$\begin{array}{cc} p(\top) = 1 & p(\bot) = 0 \\ p(a \wedge b) = p(x) \wedge p(y) & p(a \vee b) = p(a) \vee p(b) \end{array}$$

for each $a, b \in A$.

We may topologize S as a subspace of T, and then S is nothing more than

$$\mathsf{spec}\, A$$

he lattice spectrum of A. In [5] we will obtain a better understanding of this space, and that will the following construction quite obvious.

The secret ingredient of the Stone representation of a lattice is a certain choice principle used as a separation technique. Here is the version we need.

5.28 Separation Principle. *Let $a \in A$ be an element and let $I \in \mathcal{I}d\, A$ be an ideal of the frame A, and suppose $a \notin I$. Then*

$$p(a) = 1 \qquad p^{\rightarrow}(I) = \{0\}$$

for some $p \in S = \boldsymbol{Dlt}[A, 2]$.

We can write down the definition of the required morphism ι immediately.

5.29 DEFINITION. For each $U \in \mathcal{O}T = \Phi A$ let $\iota(U)$ be the subset of A given by

$$a \in \iota(U) \Longleftrightarrow \eta(a) \cap S \subseteq U$$

(for $a \in A$).

Of course, proving that this does the required job takes a little longer.

5.30 LEMMA. *For each $U \in \mathcal{O}T$ the subset $\iota(U) \subseteq A$ is an ideal of A.*

Proof. Since

$$\eta(\bot) \cap S = \emptyset$$

we have $\bot \in \iota(U)$, and trivially $\iota(U)$ is a lower section of A.

Consider $a, b \in \iota(U)$. Thus

$$\eta(a) \cap S \subseteq U \qquad \eta(b) \cap S \subseteq U$$

and we require $a \vee b \in \iota(U)$. But, for

$$p \in \eta(a \vee b) \cap S$$

we have

$$p(a) \vee p(b) = p(a \vee b) = 1$$

so that

$$p(a) = 1 \quad \text{or} \quad p(b) = 1$$

and hence

$$p \in \eta(a) \cap S \subseteq U \quad \text{or} \quad p \in \eta(b) \cap S \subseteq U$$

to give the required result. ■

trivially we have

$$\iota(\emptyset) = \emptyset \qquad \iota(\top) = A$$

and ι is monotone. Also, a few moment's thought gives

$$\iota(U) \cap \iota(V) = \iota(U \cap V)$$

for $U, V \in \mathcal{O}T$. Thus we have the following.

5.31 LEMMA. *The assignment*

$$\mathcal{O}T \xrightarrow{\ \iota\ } \mathcal{I}d\, A$$

is a $\wedge$-semilattice morphism.

Most of our work is concerned with the interaction of ι with unions in T. For that we need a preliminary observation.

5.32 LEMMA. *We have*

$$U \cap S = \bigcup \{\eta(a) \cap S \mid a \in \iota(U)\}$$

for each $U \in \mathcal{O}T$.

Proof. The inclusion '$\supseteq$' is trivial, by the construction of $\iota(U)$. Thus it suffices to verify the converse inclusion.

Consider any $p \in U \cap S$. Since $P \in U$ we have

$$p \in \eta(a_1) \cap \cdots \cap \eta(a_m) \subseteq U$$

for some $a_1, \dots, a_m \in A$. Let
$$a = a_1 \wedge \cdots \wedge a_m$$
so that
$$q(a) = q(a_1) \wedge \cdots \wedge q(a_m)$$
for each $q \in S$. This gives
$$\eta(a) \cap S = \eta(a_1) \cap \cdots \cap \eta(a_m) \cap S \subseteq U$$
so that $a \in \iota(U)$, and $p \in \eta(a) \cap S$, for the required result. ■

By construction, we know that
$$\{\eta(a) \mid a \in A\}$$
is a subbase for the topology $\mathcal{O}T$ on T. This last result shows that
$$\{\eta(a) \cap S \mid a \in A\}$$
is a *base* for the topology $\mathcal{O}S$ on the subspace S.

Consider any family $\mathcal{U} \subseteq \mathcal{O}T$. This gives us a family
$$\{\iota(U) \mid U \in \mathcal{U}\}$$
of ideals of A, and we may form the supremum
$$\bigvee \{\iota(U) \mid U \in \mathcal{U}\}$$
in $\mathcal{I}D$. How does this relate to $\iota(\bigcup \mathcal{U})$? To answer that we need the Separation Principle 5.28.

5.33 LEMMA. *Consider $\mathcal{U} \subseteq \mathcal{O}T$, and set*
$$J = \iota(\bigcup \mathcal{U}) \qquad I = \bigvee \{\iota(U) \mid U \in \mathcal{U}\}$$
to obtain two ideals J and I of A. Then $J = I$.

Proof. We have $I \subseteq J$ since $\iota(\cdot)$ is monotone.

For the converse, consider any $a \in J$ and, by way of contradiction, suppose $a \in I$. Then the Separation Principle 5.28 gives
$$p(a) = 1 \qquad p^{\rightarrow}(I) = \{0\}$$
for some $p \in S$. Since $p(a) = 1$ and $a \in J$, we have
$$p \in \eta(a) \cap S \subseteq \bigcup \mathcal{U}$$
and hence $p \in U$ for some $U \in \mathcal{U}$, But now Lemma 5.32 gives
$$p \in U \cap S = \bigcup \{\eta(b) \cap S \mid b \in \iota(U)\}$$

and hence $p(b) = 1$ for some $b \in I$. This is the contradiction. ■

With this we can quite quickly prove Theorem 5.27.

Proof of 5.27. Since the assignment $\eta = \eta_A$ is ***Frm***-epic, there is at most one such morphism $\iota = \iota_A$. Thus it suffices to show that the assignment ι constructed above is a frame morphism.

By Lemma 5.31 it remains to show

$$\iota(\bigcup \mathcal{U}) \subseteq \bigvee\{\iota(U) \mid U \in \mathcal{U}\}$$

for $\mathcal{U} \subseteq \Phi A$. But this is nothing more then Lemma 5.33.

Finally, we require

$$(\iota \circ \eta)(a) = \downarrow a$$

for each $a \in A$. To show this observe that

$$b \in (\iota \circ \eta)(a) \Longleftrightarrow \eta(b) \cap S \subseteq \eta(a)$$

and hence

$$b \leq a \Longrightarrow b \in (\iota \circ \eta)(a)$$

for each $b \in A$.

For the converse suppose $b \in (\iota \circ \eta(a)$, that is

$$\eta(b) \cap S \subseteq \eta(a)$$

by the construction of ι. If $b \not\leq a$ then a use of the Separation Principle 5.28 gives some $p \in S$ with

$$p(b) = 1 \qquad p(a) = 0$$

which, as we have just seen, can not be. ■

A close look at Lemma 5.33 show that it produces an isomorphism

$$\mathcal{O}S \cong \mathcal{I}d\,A$$

and then ι is essentially just the restriction morphism from $\mathcal{O}T$ to $\mathcal{O}S$. However, as I said, the details of that are done in [5].

References

[1] A collection of notes on my web pages with

/FRAMES/frames.html

holding the relevant documents.

[2] Overview of 'A collection of notes on frames'.

[3] The basics of frame theory.

[4] The assembly of a frame.

[5] The point space of a frame.

[6] The fundamental triangle of a space.

[7] Boolean reflections of frames.

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