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# ON DIOPHANTINE EQUATIONS OF THE FORM $(x^n - 1)(y^m - 1) = z^2$

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ABSTRACT. At the 13th Czech and Slovak Conference in Number Theory, L. Szalay showed that the Diophantine equation $(2^n - 1)(3^n - 1) = x^2$ has no solutions in positive integers $n$ and $x$. Szalay's proof used the evaluation of certain Jacobi symbols to arrive at the result. The purpose of this paper is to generalize the result of Szalay by proving that the Diophantine equation $(2^n - 1)(3^m - 1) = x^2$ has no solutions in positive integers $n, m, x$. We further discuss the solvability of the more general equation $(x^n - 1)(y^m - 1) = z^2$.

At the 13th Czech and Slovak Conference in Number Theory, L. Szalay [3] showed that the Diophantine equation

$$(2^n - 1)(3^n - 1) = x^2$$

has no solutions in positive integers $n$ and $x$. Szalay's proof used the evaluation of certain Jacobi symbols to arrive at the result. The purpose of this paper is to generalize the result of Szalay, and moreover provide a very simple solution. We conclude by making some remarks on the solvability of the more general equation given in the title.

Our main result is the following.

**THEOREM 1.** *The Diophantine equation*

$$(2^n - 1)(3^m - 1) = x^2$$

*has no solutions in positive integers $n$, $m$, and $x$.*

Proof. From the equation in the theorem it follows that there is a square-free integer $d \geq 1$, and nonzero integers $y$ and $z$, for which

$$2^n - 1 = dy^2, \qquad 3^m - 1 = dz^2.$$

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If $d = 1$, then $m$ must be odd, for otherwise $3^m$ and $z^2$ would be consecutive integers which are squares, contradicting the fact that $z$ is nonzero. But for $m$ odd, $3^m - 1 \equiv 2 \pmod 4$, which shows that $d > 1$.

From the equation $2^n - 1 = dy^2$, we have that $d$ is odd. Therefore, the equation $3^m - 1 = dz^2$ shows that $3^m - 1$ is properly divisible by an even power of $2$. It follows from the binomial theorem that $m$ is even. We now deal with the cases $n$ odd and $n$ even separately.

If $n$ is odd, then $2^n - 1 \equiv 1 \pmod 3$, and it follows that $d \equiv 1 \pmod 3$. But from the equation $3^m - 1 = dz^2$, it follows that $d \equiv 2 \pmod 3$, a contradiction. We thus conclude that $n$ is even.

Let $\varepsilon_d = T + U\sqrt{d}$ denote the minimal solution in positive integers to the Pell equation $X^2 - dY^2 = 1$, and $T_k + U_k\sqrt{d} = \left(T + U\sqrt{d}\right)^k$ for $k \geq 1$. Properties of solutions to Pell equations can be found in [2]. Since $n$ is even, we have that

$$2^{n/2} + y\sqrt{d} = T_r + U_r\sqrt{d}$$

for some odd positive integer $r$, since $T_k$ is odd for even values of $k$. This implies that $T_k$ is even for all odd positive integers $k$. From the equation $3^m - 1 = dz^2$, and the fact that $m$ is even, we conclude that

$$3^{m/2} + z\sqrt{d} = T_s + U_s\sqrt{d}$$

for some positive even integer $s$. Let $s = 2t$, then

$$3^{m/2} = T_{2t} = 2T_t^2 - 1\,,$$

from which it follows that $T_t^2 \equiv 2 \pmod 3$, which is not possible. This completes the proof of Theorem 1. $\square$

**Remarks on the general equation** $(x^n - 1)(y^m - 1) = z^2$**.**

In Theorem 1 we were able to solve the particular equation $(2^n - 1)(3^m - 1) = x^2$. It would of course be of interest to prove finiteness results for general equations of this form. In particular, it would be interesting to prove a similar result for any equation of the form

$$(x^n - 1)(y^m - 1) = z^2,$$

as there is very little known. There are examples of such equations with solutions, such as $(2^5 - 1)(5^3 - 1) = 62^2$ and $(13^4 - 1)(239^4 - 1) = 9653280^2$.

We finish the paper by raising questions about equations of this kind.

1. Let $a$ and $b$ be fixed positive integers. Under the assumption of the $abc$-conjecture, it can be shown that the equation

$$(a^n - 1)(b^n - 1) = x^2$$

has only finitely many solutions $(n, x)$. Can this result be proved without the hypothesis of the $abc$-conjecture?

2. Again with $a$ and $b$ fixed positive integers, are there infinitely many pairs of integers $(n, m)$ for which there is an integer $x$ with $(a^n-1)(b^m-1)=x^2$?

3. If $n>2$ is a fixed positive integer, are there only finitely many integers $x$ and $y$ for which $(x^n-1)(y^n-1)=z^2$ for some integer $z$? The only known case is $n=4$, which has recently been solved by J. H. E. Cohn in [1]. His result shows that the only solution for $n=4$ is $(13^4-1)(239^4-1)=9653280^2$.

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