thus having the 'right' behaviour prescribed at 1 and ∞. There is a restriction on the location of the singularity a; it is supposed that $a \notin [1, \infty)$ and in some parts of the analysis it seems necessary that $|a| < 1$, so that it is hard to relate this paper to the situation which we have taken as standard in this monograph. The series obtained are, in fact, E-type I series, as was pointed out by Erdélyi himself in his 1944 paper.

(E) *Kalnins and Miller (1991)*

This paper is concerned only with the expansion of Heun polynomials, but the treatment is radically different from that used here, and is based on group-theoretic methods and the technique of separation of variables on the n-sphere. The results include the expansion of a product of two Heun polynomials in terms of products of Jacobi polynomials.

5

Orthogonality relations

5.1 Survey

Three types of orthogonality relation are known for solutions of Heun's equation. Such relations, as we would expect, hold among solutions satisfying various boundary conditions.

(a) For Heun *functions*, relative to a pair of singularities s_1, s_2, there hold what may conveniently be called 'single' orthogonality relations, of the familiar kind associated with Sturm–Liouville eigenvalue problems. Two such Heun functions must belong to the same class (I to IV) but be distinguished by belonging, in general, to different values of the accessory parameter q. The path of integration is, generally, a Pochhammer double-loop contour about s_1, s_2, which can sometimes be reduced to a simple contour joining s_1, s_2. These will be discussed in section 5.2.

(b) For Heun *polynomials* (which, since they are also Heun functions, naturally also satisfy the relations described in (a)) there exist, also, 'double' orthogonality relations. The polynomials are of the same class (I to VIII) but are either of different degrees, and hence associated with different values of the parameter α, or of the same degree but associated with different values of the parameter q. The orthogonality applies to products of identical polynomials in two variables, and the integration is two-dimensional along paths about, or joining, two pairs of singularities. These will be discussed in section 5.3.

These relationships are of the type associated with two-parameter eigenvalue problems, described in Arscott (1964b). For Heun's equation itself, the relations were first given by Sleeman (1965).

(c) For circuit-multiplicative solutions, there are bi-orthogonal relations, in which the path of integration is a circuit surrounding a pair of singularities. These are described in Schmidt (1979, para. 2.2), and for particulars the reader is referred to that paper.

The importance of orthogonality relations lies mainly in the fact that they provide – formally, at least – the possibility of expanding a more or less arbitrary function in a series of the orthogonal functions. The single-orthogonality relations of section 5.2 lead to such expansions of a single function as a series of Heun functions, while the double-orthogonality relations of section 5.3 open the way to expansions of a function of two variables in terms of products of Heun polynomials.

5.2 Single-orthogonality relations for Heun functions

We write Heun's equation in an operator form,

$$M_z y = qy \tag{5.2.1}$$

where

$$M_z := z(z-1)(z-a)\frac{d^2}{dz^2}+[\gamma(z-1)(z-a)+\delta z(z-a)+\epsilon z(z-1)]\frac{d}{dz}+\alpha\beta z \tag{5.2.2}$$

We also write

$$w(z) := z^{\gamma-1}(z-1)^{\delta-1}(z-a)^{\epsilon-1} \tag{5.2.3}$$

for the weight function $w(z)$ which occurs in the orthogonality relations, and

$$p(z) := z^{\gamma}(z-1)^{\delta}(z-a)^{\epsilon} \tag{5.2.4}$$

The operator M_z in (5.2.2) is not self-adjoint. We therefore introduce the operator L_z given by

$$L_z = w(z)M_z \tag{5.2.5a}$$

Hence

$$L_z = \frac{d}{dz}\left(z^{\gamma}(z-1)^{\delta}(z-a)^{\epsilon}\frac{d}{dz}\right)+\alpha\beta z^{\gamma}(z-1)^{\delta-1}(z-a)^{\epsilon-1} \tag{5.2.5b}$$

and Heun's equation then takes the self-adjoint form

$$L_z y = qw(z)y \tag{5.2.6}$$

We now follow the usual procedure. Let y_1, y_2 be solutions of (5.2.6) corresponding to different values q_1, q_2 of q, so that

$$L_z y_1 = q_1 w(z) y_1, \quad L_z y_2 = q_2 w(z) y_2 \tag{5.2.7}$$

We multiply equations (5.2.7) by y_2, y_1 respectively, subtract, and integrate over a contour C, giving

$$(q_2-q_1)\int_C wy_1y_2dz = \left[p(z)(y_1y_2'-y_1'y_2)\right]_C \tag{5.2.8}$$

We now make three restrictive assumptions, two of which will shortly be removed.

(i) $a \notin [0,1]$.
(ii) $y_1(z), y_2(z)$ are Heun functions of class I relative to 0,1, corresponding to different values of q.
(iii) Re $\gamma > 0$, Re $\delta > 0$.

Then $y_i(z), y_i'(z)$ are finite at $z = 0, 1$, so if the contour C is taken as the straight line segment $[0,1]$, the term on the right hand side of (5.2.8) vanishes, and since, by hypothesis, $q_1 - q_2 \neq 0$, we have

$$\int_0^1 w(z)y_1(z)y_2(z)dz = 0 \tag{5.2.9}$$

To remove the restriction (iii), we replace the straight line contour $[0,1]$ by a Pochhammer loop contour $\mathcal{L}$ (see, for example, Whittaker and Watson 1940, para. 12.43) about the points 0,1.

The case when the $y_i(z)$ are Heun functions of class II, III, or IV (they must be of the same class) is dealt with simply by making the appropriate transformation (TII), (TIII), or (TIV) from (2.2.4), and applying the same reasoning: it is found that the same orthogonality relation holds, i.e.

$$\int_{\mathcal{L}} z^{1-\gamma}(z-1)^{1-\delta}(z-a)^{1-\epsilon}y_1(z)y_2(z)dz = 0 \tag{5.2.10}$$

where

$$y_i(z) = (0,1)Hf^{(I)}_{m_i}(q_{m_i}, z), \quad i = 1,2 \tag{5.2.11}$$

in the notation of (3.5.2).

This removes restriction (ii).

Normalization

When the functions y_1, y_2 coincide, of course, the integral no longer vanishes, and it is a matter of some interest to compute its value. Let us write

$$\theta_m := \int_{\mathcal{L}} w(z)[y_m(z)]^2 dz \tag{5.2.12}$$

where

$$y_m(z) = (0,1)Hf^{(I)}_m(q_m, z)$$

This problem was considered briefly by Lambe and Ward (1934, paras. 2.31–2.33) for the case when the Heun functions reduce to polynomials, and more fully by Erdélyi (1944, para. 9) who derived an expression for

θ_m from the expansion of $y(z)$ as an E-type II series of hypergeometric functions. Lambe and Ward showed that, with certain restrictions on the parameters, $\theta_m \neq 0$. When this property holds, we have an alternative method for *normalizing* Heun functions of a given class, associated with a given pair of singularities, in place of the convention adopted in (3.3.5), that the leading coefficient in the power-series expansion should be 1. We could, instead, multiply our Heun function by an appropriate constant so that $\theta_m = 1$. For theoretical studies, this may well be more convenient, and parallels the convention often used for other higher special functions.

Orthogonality over other intervals

The discussion above relates to a pair of Heun functions relative to the singularities 0,1. The same argument can, of course, be applied to a pair of Heun functions relative to any other two singularities, the path of integration being replaced by a corresponding Pochhammer loop contour about those singularities.

Formal expansion as a series of Heun functions

The orthogonality theorem given above leads, of course, to the formal Sturm–Liouville type of expansion of a more or less arbitrary function as a series of Heun functions. If we assume that $f(z)$ can be expressed by a series

$$f(z) = \sum_{0}^{\infty} c_m \, y_m(z) \tag{5.2.13}$$

then from (5.2.11), (5.2.12),

$$c_m = \theta_m{}^{-1} \int_{\mathcal{L}} f(z) y_m(z) w(z) dz \tag{5.2.14}$$

The validity of such expansions, except in so far as it is covered by general Sturm–Liouville theory, does not appear to have been specifically investigated.

5.3 Double-orthogonality relations for Heun polynomials

The single-orthogonality relations described in section 5.2 are of a well-known type, which occur in very many contexts. The double-orthogonality relations, which we now describe, are much less familiar, and arise essentially only in the context of two-parameter problems. We have such a situation here, when we consider Heun's equation with the requirement that a solution be a polynomial – that is to say, belonging to one of the eight types of such solutions described in section 3.6. We recall that such solutions exist only when one of the parameters α, β has one of a set of special values. According to the class, one of these parameters must be such

that the corresponding quantity in the fourth or fifth column of the table in (3.6.2) must be a negative integer or zero. For convenience, we suppose that it is α which has this property, with β being determined only through the relation $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. For the existence of such a solution, we require also that q have one of a finite number of characteristic values. Considering, in particular, polynomials of class I, we must have $\alpha = -n$, $n = 0, 1, 2, \ldots$, and there are generally $n+1$ corresponding values of q. We have thus two parameters to be specified.

The principal result is as follows:

Let $y_1 := Hp^{(X)}_{n_1,m_1}$ and $y_2 := Hp^{(X)}_{n_2,m_2}$ be Heun polynomials of the same class (X), where $X \in \{\text{I,II,}\ldots\text{,VIII}\}$; that is, Heun polynomials of the same class, but of different degrees, or else of the same degree but corresponding to different values of the accessory parameter q. For this, we must have either $n_1 \neq n_2$ or $m_1 \neq m_2$, or both, that is

$$|n_1 - n_2| + |m_1 - m_2| \neq 0$$

Let $W(s,t)$ denote the weight function

$$W(s,t) := (s-t)(st)^{\gamma-1}((s-1)(t-1))^{\delta-1}((s-a)(t-a))^{\epsilon-1} \tag{5.3.1}$$

Let

$$\mathcal{L},\ \mathcal{L}'$$

be Pochhammer loop contours about two distinct pairs of singularities

$$\{0,1\}, \qquad \{0,a\}, \qquad \{1,a\} \tag{5.3.2}$$

Then

$$\int_{\mathcal{L}} \int_{\mathcal{L}'} y_1(s) y_1(t) y_2(s) y_2(t) W(s,t) ds dt = 0 \tag{5.3.3}$$

The proof of this is somewhat lengthy, but follows the general lines used in Arscott (1964b, para. 3), and will therefore be omitted. It is given in full in Sleeman (1965).

We must now consider the value of the integral on the left of (5.3.3) when the two functions y_i coincide, say

$$y_1(z) = y_2(z) = y_{n,m}(z) = Hp^{(X)}_{n,m}(z) \tag{5.3.4}$$

Then in general the value of the integral is non-zero; care must be taken over the contours of integration, since s lies on $\mathcal{L}'$ and t on $\mathcal{L}$, and we must ensure that the factor $s - t$ which occurs in the weight function $W(s,t)$ does not vanish, except possibly on a set of measure 0. In the commonly occurring situation where $a \in (1, \infty)$, we could take $\mathcal{L}$ as a loop about 0 and 1, and $\mathcal{L}'$ a loop about 1 and a, so that $s - t$ is real and negative except at a finite number of points.

Let us write

$$\phi_{n,m} := \int_{\mathcal{L}} \int_{\mathcal{L}'} [y_{n,m}(s)]^2 [y_{n,m}(t)]^2 W(s,t) ds dt \tag{5.3.5}$$

Then another possible normalization for Heun polynomials would be to specify that the constant implicit in their construction should be so chosen that $\phi_{n,m} = 1$.

Formal expansion of a symmetric function of two variables in a double series of Heun polynomials

These results lead to the possibility of a formal expansion of a more or less arbitrary symmetric function of two variables as a double series.

Assume a function $F(s,t)$ is expressible as

$$F(s,t) = \sum_{n=0}^{\infty} \sum_{m} c_{n,m} y_{n,m}(s) y_{n,m}(t) \tag{5.3.6}$$

where summation over m is finite, depending on the number of linearly independent polynomial solutions with $\alpha = -n$; generally, the summation will run from $m = 0$ to n.

Then formally, from the relations (5.3.3), (5.3.5), the coefficients in the series are given by

$$c_{n,m} = (\phi_{n,m})^{-1} \int_{\mathcal{L}} \int_{\mathcal{L}'} F(s,t) y_{n,m}(s) y_{n,m}(t) ds dt \tag{5.3.7}$$

This analysis, it should be stressed, is only formal, and space prevents any discussion of the general validity of such an expansion.

6

Integral equations and integral relations

6.1 Survey

Except in some trivial cases, no example has been given of a solution of Heun's equation expressed in the form of a definite integral or contour integral involving only functions which are, in some sense, simpler. It may be reasonably conjectured that no such expressions exist. In place of such formulae, there are integral equations satisfied by solutions of Heun's equation, and integral relationships expressing one solution in terms of another.

There are two kinds of relations to be considered:

(a) Linear relations, including Fredholm integral equations of the second kind. The key fact here is that if $y(z)$ is a solution of Heun's equation, satisfying certain conditions, while the *kernel* $K(z,t)$ and the contour C are appropriately chosen, the function

$$Y(z) := \int_C K(z,t) y(t) w(t) dt \tag{6.1.1}$$

is also a solution of Heun's equation (K, the kernel, has to be chosen to satisfy a certain partial differential equation, while w is a weight function). If K and C are chosen so that $Y(z)$ is necessarily a multiple of $y(z)$ then (6.1.1) becomes an integral equation.

(b) Non-linear relations, involving the integral of a product of two solutions in different variables. These are sometimes known as Malurkar-type integral relations, after their first occurrence in connection with ellipsoidal wave functions.

The key result in this connection is that if $y(z)$ is a solution of Heun's equation, the function $H(z,s,t)$ a solution of a certain partial differential equation in z, s, t, $w(s,t)$ a weight function, and the contours C_1, C_2 appropriately chosen, then

$$Y(z) := \int_{C_1} \int_{C_2} H(z,s,t) y(s) y(t) w(s,t) ds dt \tag{6.1.2}$$

is also a solution of Heun's equation. With appropriate choices of H, C_1, C_2, $Y(z)$ may be a multiple of $y(z)$, in which case we have a non-linear integral equation for $y(z)$.

The first work on integral equations for Heun functions was that of Lambe and Ward (1934), on equations of type (6.1.1), considering only Heun polynomials. This was developed by Erdélyi (1942b) who extended the theory to solutions which are not polynomials. Integral relations, as distinct from integral equations, were not considered. This topic is discussed in section 6.2.

Integral equations of type (6.1.2) were considered by Sleeman (1968), and this work will be described in section 6.3.

Two papers dealing with these topics, each in a wider context, are (a) Arscott (1964b), which is concerned with general two-parameter eigenvalue problems, and (b) Schmidt and Wolf (1979) which utilizes the concept of simultaneous separability in different coordinate systems.

Sleeman (1969) considered a slightly different topic, and obtained representations for solutions of Heun's equation in the form of Barnes-type integrals (compare Whittaker and Watson 1940, 14.5, 16.4) in which the integrand is a somewhat complicated series involving the coefficients in the power series (3.4.1).

6.2 Linear integral equations and relations

We write Heun's equation (1.1.1) as

$$M_z y = qy \tag{6.2.1}$$

where M_z is as defined in (5.2.2), but since it will here be applied to functions of more than one variable, it is convenient to regard it as a partial differential operator

$$M_z := z(z-1)(z-a)\frac{\partial^2}{\partial z^2}+[\gamma(z-1)(z-a)+\delta z(z-a)+\epsilon z(z-1)]\frac{\partial}{\partial z}+\alpha\beta z \tag{6.2.2}$$

We also write

$$w(z) = z^{\gamma-1}(z-1)^{\delta-1}(z-a)^{\epsilon-1} \tag{6.2.3}$$

$$p(z) = z^{\gamma}(z-1)^{\delta}(z-a)^{\epsilon} \tag{6.2.4}$$

as in section 5.2.

Then the principal result is as follows. Let

(i) $y(z)$ be a solution of Heun's equation (6.2.1),

(ii) $K(z,t)$ satisfy the partial differential equation

$$(M_z - M_t)K(z,t) = 0 \tag{6.2.5}$$

(iii) the path of integration C be such that the 'integrated part' vanishes, i.e.

$$\left[p(t)\left(\frac{\partial K}{\partial t}y(t) - K\frac{dy}{dt}\right)\right]_C = 0 \tag{6.2.6}$$

Then $Y(z)$, defined by

$$Y(z) := \int_C K(z,t)y(t)w(t)dt \tag{6.2.7}$$

is a solution of Heun's equation (1.1.1), provided the integral exists and, if singular, converges uniformly with respect to z in an appropriate domain.

The three (interconnected) problems which now arise are

(i) finding solutions of the partial differential equation (6.2.5),

(ii) finding a suitable contour C such that (6.2.6) is satisfied,

(iii) identifying the resulting function $Y(z)$.

Problem (iii) must be explained further. Essentially, the character of $Y(z)$ will depend on the character of $K(z,t)$ as a function of z. This may be shown, for instance, by its being a polynomial in z, or by corresponding to a particular local solution. In this way it may often be established that $Y(z)$ is a (non-zero) multiple of $y(z)$, in which case (6.2.7) becomes a linear integral equation for $y(z)$.

On the other hand, it may be that $Y(z)$ is a different solution, possibly valid in a different region of the z-plane from $y(z)$, thus relating two distinct solutions of Heun's equation, so that we have an integral relation. It should be noted that $K(z,t)$ is generally a simpler function of z than is $y(z)$, so that such integral relations may be useful, for instance, in finding asymptotic behaviour of $Y(z)$ either with respect to the variable z or with respect to a variable parameter. This possibility has been extensively exploited for solutions, for example of Mathieu's and Lamé's equations, but does not seem to have been significantly explored for Heun's equation.

Finally, we must not ignore the possibility that $Y(z)$ is the identically zero solution. Care must be taken to exclude this possibility before claiming the practical validity of an integral equation or relation.

The papers of Lambe and Ward and of Erdélyi are concerned primarily with the problem of finding suitable kernels $K(z,t)$ satisfying (6.2.5).

It should be noted that (6.2.5) does not involve the accessory parameter q, so that although it is, of course, satisfied by a product $K(z,t) = y(z)y(t)$, where $y(z)$ is a solution of (6.2.1), the power of the method consists in finding simpler $K(z,t)$.

If we are able, in some way, to identify $Y(z)$ as a multiple of $y(z)$, say $Y(z) = \lambda y(z)$, then this fact can by expressed by asserting that $y(z)$ is an eigenfunction of the integral equation

$$\phi(z) = \lambda \int_C K(z,t)\phi(t)w(t)dt \tag{6.2.8}$$

λ being the corresponding eigenvalue. Naturally, the value of λ depends on the particular solution $y(z)$; let us therefore write more specifically

$$y_m(z) = \lambda_m \int_C K(z,t)y_m(t)w(t)dt \tag{6.2.9}$$

Evaluation of the eigenvalue λ is important, but seems to have been attempted only in a few cases. If, for instance, $y(z)$ is a Heun function of class I or class III, so that $y(0)$ is finite and calculable, then we may set $z = 0$ in (6.2.8) and obtain

$$(\lambda_m)^{-1}y_m(0) = \int_C K(0,t)y_m(t)w(t)dt \tag{6.2.10}$$

The search for kernels

We return now to the problem of finding suitable kernels $K(z,t)$ satisfying (6.2.5).

Both Lambe and Ward and Erdélyi transform to new variables θ, ϕ (in effect, polar coordinates on the unit sphere) given by

$$\cos\theta = \left(\frac{zt}{a}\right)^{\frac{1}{2}} \tag{6.2.11a}$$

$$\sin\theta\cos\phi = \left(\frac{(z-a)(t-a)}{a(1-a)}\right)^{\frac{1}{2}} \tag{6.2.11b}$$

$$\sin\theta\sin\phi = \left(\frac{(z-1)(t-1)}{1-a}\right)^{\frac{1}{2}} \tag{6.2.11c}$$

The partial differential equation to be satisfied by K then becomes, in the θ, ϕ variables,

$$\sin^2\theta\left(\frac{\partial^2 K}{\partial\theta^2} + [(1-2\gamma)\tan\theta + 2(\delta+\epsilon-\tfrac{1}{2})\cot\theta]\frac{\partial K}{\partial\theta} - 4\alpha\beta K\right)$$
$$+\frac{\partial^2 K}{\partial\phi^2} + \Big[(1-2\delta)\cot\phi - (1-2\epsilon)\tan\phi\Big]\frac{\partial K}{\partial\phi} = 0 \tag{6.2.12}$$

This equation is separable, and solutions may be obtained in the form $K = \Theta(\theta)\Phi(\phi)$ where Θ and Φ satisfy equations of the Riemann P form,

and hence are expressible in terms of hypergeometric functions. The occurrence of a separation parameter leads to a wide choice of possible kernels, including those obtainable by summation of different kernels or even integration with respect to the separation parameter. The arguments of the hypergeometric functions are

$$\cos^2\theta = \frac{zt}{a}, \quad \cos^2\phi = \frac{(z-a)(t-a)}{(1-a)(zt-a)} \tag{6.2.13}$$

Lambe and Ward consider kernels which are hypergeometric functions of the argument $zt/a = \cos^2\theta$; they are concerned principally with integral equations satisfied by Heun polynomials of class I, for which $K(z,t) = {}_2F_1(\alpha,\beta;\gamma;zt/a)$, but also give nuclei for equations satisfied by polynomials of the other classes.

Erdélyi obtains a more general class of nuclei, in the form of a product of a hypergeometric function of $\cos^2\theta$ and another hypergeometric function of $\cos^2\phi$. More generally, kernels can be obtained by a sum of such functions or even an integral over the separation parameter which arises in this representation from the process of separating the equation (6.2.12). Erdélyi shows that the integral equations so obtained are satisfied not only by Heun polynomials but by Heun functions also.

Erdélyi (1944, para. 10) gives an interesting application of this, by showing that when an E-type II series for a Heun function is substituted into the right hand side of a certain integral equation, the left hand side yields an E-type I series for the same function.

Lambe and Ward devote attention also to a number of related equations, i.e. special and confluent cases of Heun's equation, namely (i) the Lamé equation, when the kernels become Legendre functions, (ii) the associated Lamé equation, when they become Gegenbauer functions, (iii) the confluent Heun equation, when they become confluent hypergeometric functions, and finally (iv) a generalized Mathieu equation, for which the kernels are Hankel functions.

Integral relations

Our discussion above has concentrated on the situation where the $Y(z)$ of (6.2.7) is a multiple of $y(z)$, resulting in an integral equation. This happens, roughly speaking, when $K(z,t)$, regarded as a function of z, has the same kind of behaviour as $y(z)$. It is perfectly possible, however, to find solutions of (6.2.5) where $K(z,t)$ has different behaviour, so that $Y(z)$, while still a solution of the differential equation, is not the same as $y(z)$. Another possibility for obtaining such relations is to alter the contour C suitably.

An integral relation cannot, of course, be expressed as precisely as an integral equation, since it depends on the identification of the particular solution $Y(z)$ which the integral represents. We can only write the analogue of (6.2.10) in a rather general form

$$Y_m(z) = \mu_m \int_C K(z,t) y_m(t) w(t) dt \tag{6.2.14}$$

for some constant μ_m.

Very little appears to have been done on the exploitation of these results, though many instances are known, in other areas of higher special functions, where formulae of this kind give valuable integral representations of solutions in terms of other (generally simpler) solutions, and also of asymptotic expressions for particular solutions, with respect to either a large variable or a large parameter. This is one of the areas of Heun's equation in which further investigations may be very fruitful.

Formal bilinear development of the kernel

An extremely useful application of integral equations and relationships of the types described above is to the expansion of a kernel $K(z,t)$ as a series of products of the form $f(z)g(t)$.

Consider, first, the situation where $K(z,t)$ is a symmetric function of z and t, and where the relationship is an integral equation of the form (6.2.9).

Suppose also that $K(z,t)$, regarded as a function of z, is such that it can be expanded as a series

$$K(z,t) = \sum_m c_m(t) y_m(z) \tag{6.2.15}$$

Then we can apply the result of (5.2.14), use (5.2.12), (6.2.15), and obtain

$$c_m(t) = (\theta_m \lambda_m)^{-1} y_m(t)$$

hence formally

$$K(z,t) = \sum_m (\theta_m \lambda_m)^{-1} y_m(z) y_m(t) \tag{6.2.16}$$

The other situation, in which $K(z,t)$ is not symmetric, and the relationship is an integral relation of the form (6.2.14), gives a similar expansion of $K(z,t)$ in a series of the products $Y_m(z)y_m(t)$.

6.3 Non-linear integral equations and relations

The key paper here is that of Sleeman (1969b), which gives the principal idea but is incorrect in some details.

The basic theorem is the following:

Let

(i) $y(z)$ be a solution of Heun's equation (1.1.1),

(ii) $H(z,s,t)$ be a solution of the partial differential equation

$$(t-z)M_s(H) + (z-s)M_t(H) + (s-t)M_z(H) = 0 \tag{6.3.1}$$

where M is the operator defined in (5.2.2),

(iii) C_1 and C_2 be suitable paths in the complex s,t planes such that both the 'integrated parts'

$$\left[s^\gamma (s-1)^\delta (s-a)^\epsilon \left(y(s)\frac{\partial H}{\partial s} - H\frac{dy(s)}{ds} \right) \right]_{C_1} \tag{6.3.2a}$$

and

$$\left[t^\gamma (t-1)^\delta (t-a)^\epsilon \left(y(t)\frac{\partial H}{\partial t} - H\frac{dy(t)}{dt} \right) \right]_{C_2} \tag{6.3.2b}$$

vanish,

(iv) the weight function $W(s,t)$ be defined as in (5.3.1),

(v) the function $Y(z)$ defined by

$$Y(z) := \int_{C_1} \int_{C_2} H(z,s,t) y(s) y(t) W(s,t) ds dt \tag{6.3.3}$$

exist and, if the integral is singular, let it converge uniformly with respect to z when z, s, t lie in appropriate regions.

Then $Y(z)$ is a solution of Heun's equation.

As for the linear integral equations of section 6.2, there are various possibilities for $Y(z)$. It may be a non-zero multiple of $y(z)$, in which case we have a non-linear integral equation for y. It may be a different solution of Heun's equation, giving us thereby an integral relationship between two solutions. Finally, it may be the trivial zero solution. These possibilities depend largely, though not completely, on the character of $H(z,s,t)$ as a function of z.

The search for kernels

The problem now is to obtain solutions $H(z,s,t)$ of the partial differential equation (6.3.1). This is made easier by a change of variables; we introduce new variables u, v, w given by

$$u = \frac{(stz)^{\frac{1}{2}}}{a} \tag{6.3.4a}$$

$$v = \left[\frac{(s-a)(t-a)(z-a)}{a(1-a)} \right]^{\frac{1}{2}} \tag{6.3.4b}$$

$$w = \left(\frac{i}{a} \right) \left[\frac{(s-1)(t-1)(z-1)}{1-a} \right]^{\frac{1}{2}} \tag{6.3.4c}$$

In terms of these variables, equation (6.3.1) becomes

$$H_{uu} + H_{vv} + H_{ww} + (2\gamma - 1)\frac{H_u}{u} + (2\delta - 1)\frac{H_v}{v} + (2\epsilon - 1)\frac{H_w}{w} = 0 \quad (6.3.5)$$

(Sleeman (1992) corrects his paper at this point: his previous work is valid only in the case when $\delta = \epsilon = \gamma$.)

There are two possibilities immediately suggested, each of which can be made to yield a large number of possible nuclei H.

(i) We may proceed, as Sleeman does, on similar lines to those used by Lambe and Ward (1934) and Erdélyi (1942b), namely to make the further transformations

$$u = r\cos\theta, \qquad v = r\sin\theta\sin\phi, \qquad w = r\sin\theta\cos\phi \qquad (6.3.6)$$

and obtain a partial differential equation which may be separated in the r, θ, ϕ variables. This leads to possible nuclei involving the product of a power of r, a hypergeometric function of $\cos^2\theta$, and a hypergeometric function of $\cos^2\phi$.

Such nuclei can clearly be constructed on lines indicated by Sleeman's work, but do not appear to have been explicitly obtained.

(ii) It is possible to separate the equation (6.3.5) directly in u, v, w coordinates, resulting in three equations each of which can be solved in terms of Bessel functions. This possibility has not yet been explored.

It should be noted that the treatment here is a further example of the technique of 'simultaneous separability' explained by Leitner and Meixner (1960) and more fully by Schmidt and Wolf (1979). In effect, the z, s, t coordinates are Cartesian, the u, v, w are ellipsoidal, and the r, θ, ϕ are spherical coordinates.

Formal expansion of the kernel

The possibility now exists of obtaining a formal expansion of a kernel $H(z, s, t)$ as a series of products $f_1(z)f_2(s)f_3(t)$, particularly in the case where $H(z, s, t)$ is symmetric in the two variables s, t. We may regard $H(z, s, t)$ as a function of the two variables s, t, assume an expansion as a double sum

$$H(z, s, t) = \sum_n \sum_m c_{m,n}(z) y_{m,n}(s) y_{m,n}(t)$$

and, as in (5.3.6), and then use (5.3.3), (5.3.7), and the integral equation or relation to obtain the $c_{m,n}(z)$ as appropriate solutions of Heun's equation. With regret, we have to leave this topic as an interesting and possibly very valuable area for further investigation.

Felix M. Arscott

7

Appendix

A – A glossary of technical terms

In this appendix, we give a brief explanation of a number of technical terms from the general theory of linear differential equation which are used in the body of the text.

We refer to the second-order linear homogeneous equation

$$\frac{d^2w}{dz^2} + p(z)\,\frac{dw}{dz} + q(z)w = 0 \qquad (A.1)$$

where w, z are regarded as complex variables, and $p(z), q(z)$ are meromorphic functions.

1. *Ordinary points and singularities*

A point z_0 of the plane is said to be an *ordinary point* of the equation (A.1) if $p(z)$ and $q(z)$ are both regular at z_0.

Any other point is a *singularity* of the equation.

2. *Regular and irregular singularities*

Let z_0 be a singularity (so that at least one of $p(z), q(z)$ has a singularity at z_0). Then if, at z_0,

$p(z)$ is regular or has a pole of order 1, and

$q(z)$ is regular or has a pole of order not exceeding 2,

we say the singularity at z_0 is *regular*. Otherwise it is *irregular*. Thus, at a regular singularity,

$$\lim_{z\to z_0}(z - z_0)p(z) = A, \qquad \lim_{z\to z_0}(z - z_0)^2 q(z) = B \qquad (A.2)$$

both exist.

3. *Indicial equation at a regular singularity: exponents*

The equation

$$\rho^2 + (A - 1)\rho + B = 0 \qquad (A.3)$$

is called the *indicial equation* at the point z_0. Its roots ρ_1, ρ_2 are the *characteristic exponents* at z_0.