A Diophantine Equation of Degree Five

Proposed by Nikolai Osipov, Siberian Federal University, Krasnoyarsk, Russia. Solve the equation

$$xy^3 + y^2 - x^5 - 1 = 0$$

in positive integers.

Solution by the proposer. We can exploit the elementary method proposed in the paper [2] (see the examples from § 1). In fact, this method is intended to be used for solving some diophantine equations of degree four, but we can slightly modify it.

Consider the equation

$$xy^3 + y^2 - x^5 - 1 = 0 \tag{1}$$

for positive integers x and y. Clearly, if $x = 1$ then $y = 1$. Suppose that $x \geqslant 2$. For the function $y = \Psi(x)$ implicitly defined by (1), one can obtain the estimates

$$x^{4/3} - 1 < \Psi(x) < x^{4/3} \tag{2}$$

for arbitrary $x \geqslant 2$. Indeed, consider the function

$$F(y) = xy^3 + y^2 - x^5 - 1$$

on the interval $y > 0$. It can be easily checked that

$$F(x^{4/3} - 1) < 0, \quad F(x^{4/3}) > 0.$$

The estimates (2) are valid since, for every $x \geqslant 2$, the function $F(y)$ has a unique root on the given interval.

The equation (1) can be rewritten as $y^2 - 1 = lx$ with $l = x^4 - y^3$. Then

$$l \equiv -y \pmod{x}$$

because $y^2 \equiv 1 \pmod{x}$. Let $l + y = mx$ with some integer m. We have

$$mx = l + y = x^4 - y(y^2 - 1) = x^4 - lxy.$$

Hence, $m = x^3 - ly \equiv y^2 \equiv 1 \pmod{x}$. Consequently, the number

$$k = \frac{m-1}{x} = \frac{l+y-x}{x^2} = \frac{y^2 + xy - x^2 - 1}{x^3}$$

must be integer. But from (2), it follows that $\Psi(x) \sim x^{4/3}$ as $x \to \infty$. Therefore, $k \sim x^{-1/3}$ as $x \to \infty$. Thus, k is close to zero for large x. Because k is an integer, we must conclude that $k = 0$. In other words, we have

$$y^2 + xy - x^2 - 1 = 0 \tag{3}$$

if x is sufficiently large. It remains to give an explicit lower bound M for such x's. Next, we would need the following: 1) solve the system of equations (1) and (3) under the assumption $x > M$, and 2) consider those x's which satisfy $2 \leqslant x \leqslant M$.

Fortunately, there is a simpler way: it is more convenient to prove the inequalities

$$0 < \frac{\Psi(x)^2 + x\Psi(x) - x^2 - 1}{x^3} < 1 \tag{4}$$

for every $x \geqslant 2$. This is easy to implement due to the obtained estimates (2).

Finally, the inequalities (4) mean that the equation (1) has no any solutions (x, y) with $x \geqslant 2$. Thus, $(x, y) = (1, 1)$ is the unique solution. ■

Remark. The proposed diophantine equation can be solved using the well-known *Runge's method* (see, e.g., [1] and, especially, [3]). The original idea of C. Runge is to construct an *auxiliary equation*

$$g(x, y) = 0 \tag{5}$$

with the following property: each integer solution (x, y) of (1) with sufficiently large x is also a solution of (5). For the desired polynomial $g(x, y) \in \mathbb{Z}[x, y]$, we must have

$$\lim_{x\to\infty} g(x, y) = \lim_{x\to\infty} g(x, \Psi(x)) = 0,$$

where $y = \Psi(x)$ is the implicit function defined by (1). For this purpose, we can use the *Puiseux series* for $y = \Psi(x)$ at $x = \infty$, namely

$$x^{4/3} - \frac{1}{3x} + \frac{1}{9x^{10/3}} + \frac{1}{3x^{11/3}} + \dots$$

Using the *method of undetermined coefficients*, we get the polynomial

$$g(x, y) = (y^2 + 1)x^4 - yx^3 + x^2 - y^5 - y^3$$

for which we have

$$g(x, \Psi(x)) = \frac{1}{x^{1/3}} + O\left(\frac{1}{x^{2/3}}\right), \quad x \to \infty.$$

Dividing $g(x, y)$ by $f(x, y) = xy^3 + y^2 - x^5 - 1$ with remainder (as the polynomials only on y), we obtain

$$\frac{y^2 + xy - x^2 - 1}{x^3}$$

as the remainder—this is exactly the expression that we found for the number k via the elementary approach presented above.

REFERENCES

1. L.J. Mordell, *Diophantine equations*, Academic Press Inc., London, 1969.
2. N.N. Osipov, *Runge's method for the equations of fourth degree: an elementary approach*, Matematicheskoe Prosveshchenie, **19** (2015), 178–198. (In Russian)
3. C. Runge, *Ueber ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen*, J. reine und angew. Math., **100** (1887), 425–435.